Curvature homogeneous metrics and quadratic curvature functionals

A pseudo-Riemannian manifold (M, g) is k-curvature homogeneous if, for each pair of points $p, q \in M$, there exists a linear isometry $\Phi_{pq} : T_pM \to T_qM$ satisfying $\Phi_{pq}^* \nabla^i R_q = \nabla^i R_p$ for all $0 \leq i \leq k$. If $k = 0$, we say that the manifold is curvature homogeneous.

Clearly, homogeneous manifolds are curvature homogeneous but the converse is not true. Any three-dimensional 1-curvature homogeneous Riemannian manifold is locally homogeneous, but there are two strictly 1-curvature homogeneous Lorentzian three-manifold.

The purpose of this lecture is twofold. Firtly, we present the classification of 1-curvature homogeneous Lorentzian three-manifolds which are critical for some quadratic curvature functional. Secondly, we consider some special cases of curvature homogeneous manifolds and analyze the criticality of their metrics with respect to different curvature functionals. More specifically, we focus on those which are locally conformally flat or semi-symmetric, i.e., their curvature tensor at each point is that of a symmetric space. A remarkable fact is that the underlying structure in all solutions is either a Brinkmann wave or a Kundt spacetime. In particular, this classification provides non-Einstein metrics which are critical for all quadratic curvature invariants.