Homotopic distance (I and II)

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In these two talks we will show that well known invariants like Lusternik-Schnirelmann category and Farber's topological complexity are particular cases of a more general notion, that we call *homotopic distance* between maps.

Given two continuous maps $f, g\colon X \to Y$, the homotopic distance $\mathrm{D}(f, g)$ between them is the infimum of the integers $n \geq 0$ for which there exists an open covering $\{U_0, \ldots, U_n\}$ of X such that the restrictions of f and g to U_j are homotopic maps, for all $j = 0, \ldots, n$.

For instance, if X is path-connected, with base-point x_0, the LS-category $\operatorname{cat} X$ is the distance between the identity and the constant map x_0. Also, it is the distance between the axis maps $i_1, i_2\colon X \to X \times X$.

On the other side, the topological complexity $\mathrm{TC}(X)$ is the distance between the two projections $p_1, p_2\colon X \times X \to X$. Also, it is the distance between the identity of $X \times X$ and the flip map.

As a consequence, several properties of those invariants can be proved in a unified way and new results arise.

Part I (David Mosquera) ([1]) is devoted to providing the basic definitions and examples of homotopic distance, as well as proving several properties, namely its behaviour under compositions and products, and its homotopical invariance. Then several properties of cat and TC can be proved in a unified way as particular cases of the inequalities $D(f, g) \leq \operatorname{cat}(X)$ and $D(f, g) \leq \mathrm{TC}(Y)$. One striking fact of the homotopic distance is that it verifies the triangular inequality, at least for normal spaces.

Finally we will discuss several cohomological lower bounds for $\mathrm{D}(f, g)$.

In Part II (Enrique Macías)([2]) we will prove that the homotopic distance between two maps defined on a manifold is bounded above by the sum of their subspace distances on the critical submanifolds of any Morse-Bott function. This generalizes the classical Lusternik-Schnirelmann theorem (for

Morse functions) and a similar result by Farber for the topological complexity.

Analogously, we prove that, for analytic manifolds, the homotopic distance is bounded by the sum of the subspace distances on any submanifold and its cut locus.

As an application, we will show how navigation functions can be used to solve the following generalized motion planning problem: for two continuous maps $f, g\colon X \to Y$, given an arbitrary point $x \in X$, find a continuous path $s(x)$ in Y, joining $f(x)$ and $g(x)$, in such a way that the path $s(x)$ depends continuously on x. This situation includes as particular cases the *topological complexity of the work map* by Farber and Murillo-Wu; the *topological complexity of a map* by Pavesic; and the *weak category* of Yokoi.

Finally, we shall discuss several extensions of homotopic distance to other settings like simplicial complexes or small categories [3].

References

[1] E. Macías-Virgós and D. Mosquera-Lois, *Math. Proc. Camb. Philos. Soc.* 172, No. 1, 73–93 (2022).

[2] Macías-Virgós, Enrique; Mosquera-Lois, David; Pereira-Sáez, María José, *Mediterr. J. Math.* 19, No. 6, Paper No. 258, 20 p. (2022).

[3] E. Macías-Virgós and D. Mosquera-Lois, Homotopic distance between functors. *J. Homotopy Relat. Struct.* 15, No. 3–4, 537–555 (2020).