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Semigroups of transformations with restricted range or kernel

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Let X be a non-empty set and let $T(X)$ denote the semigroup (under composition) of all total transformations of X. For each α in $T(X)$, we let $X\alpha = \operatorname{ran}\alpha$ denote the *range* of α and we define the *rank* of α to be $r(\alpha) = |\operatorname{ran}\alpha|$. If $\emptyset \neq Y \subseteq X$, we write

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}.$$

Clearly $T(X,Y)$ is a subsemigroup of $T(X)$, and if $Y = X$ then $T(X,Y) = T(X)$. Also, if $|Y| = 1$ then $T(X,Y)$ contains exactly one element: namely, the constant map with range Y.

This semigroup is almost never isomorphic to $T(Z)$ for any set Z and so it is worthy of study in its own right. In 1975, Symons described all automorphisms of $T(X,Y)$. Several years later, its regular elements were characterised by Nenthein, Youngkhong and Kemprasit. Also, in 2008, Sanwong and Sommanee determined the largest regular subsemigroup of $T(X,Y)$ and using this, they described Green's relations on $T(X,Y)$. More recently, Sanwong, Singha and Sullivan characterised all maximal and minimal congruences on $T(X,Y)$.

During this talk, we will describe the ideal structure of $T(X,Y)$ and show how certain algebraic semigroups can be 'anti-embedded' in some $T(X,Y)$.

This work completes a project in which Green's relations and ideals are determined for semigroups that appear to be related but are almost never isomorphic or anti-isomorphic. Namely, the semigroups $T(X,Y)$ and its linear analogue $T(V,W)$, as well as the semigroups

$$\begin{aligned} K(V,W) &= \{\alpha \in T(V) : W \subseteq \ker\alpha\} \\ E(X,\sigma) &= \{\alpha \in T(X) : \sigma \subseteq \pi_\alpha\}, \end{aligned}$$

where W is a subspace of a vector space V, $T(V)$ denotes the semigroup (under composition) of all linear transformations from V into V, σ is a fixed equivalence on X and $\pi_\alpha = \alpha \circ \alpha^{-1}$.

[joint work with R. P. Sullivan]