PROOF THEORY: From arithmetic to set theory

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Plan of the Talks

- **First Lecture**
  1. **From Hilbert to Gentzen.**
  2. **Gentzen's Hauptsatz and applications**
  3. **The general form of ordinal analysis**
- **Second Lecture:**
  1. **Proof theory of (sub)systems of second order arithmetic.**
  2. **Applications of Ordinal Analysis**
- **Third Lecture: Proof theory of systems of set theory.**

The Origins of Proof Theory (Beweistheorie)

- Hilbert's second problem (1900): Consistency of Analysis
- Hilbert's Programme (1922,1925)

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- In 1917/18 Hilbert flirted again with logicism. Presented analysis in ramified type theory with the axiom of reducibility.
- Hilbert's finitist consistency program only emerged in the winter term 1921/22.

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- To carry out this task, Hilbert inaugurated a new mathematical discipline: **Beweistheorie** (**Proof Theory**).

- In Hilbert's Proof Theory, proofs become mathematical objects sui generis.

Ackermann's Dissertation 1925

Consistency proof for a second-order version of **Primitive Recursive Arithmetic**.

Uses a finitistic version of **transfinite induction** up to the ordinal $\omega^{\omega^{\omega}}$.

Gentzen's Result

- **Gerhard Gentzen** showed that transfinite induction up to the ordinal

$$\varepsilon_0 \;=\; \text{sup}\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\} \;=\; \text{least}\ \alpha.\ \omega^\alpha = \alpha$$

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- Gentzen's applied transfinite induction up to ε_0 solely to **primitive recursive predicates** and besides that his proof used only **finitistically justified means.**

Gentzen's Result in Detail

- $$\mathbf{F} + \text{PR-}\mathbf{TI}(\varepsilon_0) \vdash \mathbf{Con}(\mathbf{PA}),$$

  where **F** signifies a theory that is acceptable in finitism (e.g. $\mathbf{F} = \mathbf{PRA}$ = Primitive Recursive Arithmetic) and PR-**TI**(ε_0) stands for transfinite induction up to ε_0 for primitive recursive predicates.

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- Gentzen also showed that his result is best possible: **PA** proves transfinite induction up to α for arithmetic predicates for any $\alpha < \varepsilon_0$.

The Compelling Picture

The **non-finitist** part of **PA** is encapsulated in PR-**TI**(ε_0) and therefore "measured" by ε_0, thereby tempting one to adopt the following definition of **proof-theoretic ordinal** of a theory T:

$$|T|_{Con} \quad = \quad \text{least } \alpha.\ \textbf{PRA} + \text{PR-}\textbf{TI}(\alpha) \vdash \text{Con}(T).$$

The supremum of the provable ordinals

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- The supremum of the provable well-orderings of **T**:

$$|\mathbf{T}|_{\mathrm{sup}} := \sup \{\alpha : \alpha \text{ provably computable in } \mathbf{T}\}.$$

Ordinal Structures

We are interested in representing specific ordinals α as relations on $\mathbb{N}$.

Natural ordinal representation systems are frequently derived from structures of the form

$$\mathfrak{A} \quad = \quad \langle \alpha, f_1, \ldots, f_n, <_\alpha \rangle$$

where α is an ordinal, $<_\alpha$ is the ordering of ordinals restricted to elements of α and the f_i are functions

$$f_i : \underbrace{\alpha \times \cdots \times \alpha}_{k_i \text{ times}} \longrightarrow \alpha$$

for some natural number k_i.

Ordinal Representation Systems

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is a **computable** (or **recursive**) **representation** of $\mathfrak{A} = \langle \alpha, f_1, \ldots, f_n, <_\alpha \rangle$ if the following conditions hold:

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1. $A \subseteq \mathbb{N}$ and A is a computable set.
2. $\prec$ is a computable total ordering on A and the functions g_i are computable.
3. $\mathfrak{A} \cong \mathbb{A}$, i.e. the two structures are isomorphic.

Cantor's Representation of Ordinals

Theorem (Cantor, 1897) For every ordinal $\beta > 0$ there exist unique ordinals $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ such that

$$\beta = \omega^{\beta_0} + \ldots + \omega^{\beta_n}. \tag{1}$$

The representation of β in (1) is called the **Cantor normal form**.

We shall write $\beta =_{CNF} \omega^{\beta_1} + \cdots \omega^{\beta_n}$ to convey that $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_k$.

A Representation for ε_0

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- $\beta < \varepsilon_0$ has a Cantor normal form with exponents $\beta_i < \beta$ and these exponents have Cantor normal forms with yet again smaller exponents. As this process must terminate, ordinals $< \varepsilon_0$ can be coded by natural numbers.

Coding ε_0 in $\mathbb{N}$

Define a function

$$\ulcorner . \urcorner : \varepsilon_0 \longrightarrow \mathbb{N}$$

by

$$\ulcorner \delta \urcorner = \begin{cases} 0 & \text{if } \delta = 0 \\ \langle \ulcorner \delta_1 \urcorner, \ldots, \ulcorner \delta_n \urcorner \rangle & \text{if } \delta =_{CNF} \omega^{\delta_1} + \cdots \omega^{\delta_n} \end{cases}$$

where $\langle k_1, \cdots, k_n \rangle := 2^{k_1+1} \cdot \ldots \cdot p_n^{k_n+1}$ with p_i being the ith prime number (or any other coding of tuples). Further define

$$\begin{aligned} A_0 &:= \mathbf{ran}(\ulcorner . \urcorner) \\ \ulcorner \delta \urcorner \prec \ulcorner \beta \urcorner &:\Leftrightarrow \delta < \beta \\ \ulcorner \delta \urcorner \hat{+} \ulcorner \beta \urcorner &:= \ulcorner \delta + \beta \urcorner \\ \ulcorner \delta \urcorner \hat{\cdot} \ulcorner \beta \urcorner &:= \ulcorner \delta \cdot \beta \urcorner \\ \hat{\omega}^{\ulcorner \delta \urcorner} &:= \ulcorner \omega^{\delta} \urcorner . \end{aligned}$$

Coding ε_0 *in* $\mathbb{N}$

Then

$$\langle \varepsilon_0, +, \cdot, \delta \mapsto \omega^\delta, < \rangle \quad \cong \quad \langle A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, \prec \rangle.$$

$A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, \prec$ are recursive, in point of fact, they are all elementary recursive.

Transfinite Induction

- **TI**$(A, \prec)$ is the schema

$$\forall n \in A\, [\forall k \prec n\, P(k) \ \rightarrow \ P(n)] \ \rightarrow \ \forall n \in A\, P(n)$$

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- For $\alpha \in A$ let $\prec_\alpha$ be $\prec$ restricted to $A_\alpha := \{\beta \in A \mid \beta \prec \alpha\}$.

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- Every ordinal analysis of a classical or intuitionistic theory **T** that has ever appeared in the literature provides an EORS $\langle A, \lhd, \ldots \rangle$ such that **T** is finitistically reducible to

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- $|\mathbf{T}|_{\sup} = |\lhd|$.

Ordinally Informative Proof Theory

The two main strands of research are:

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- **Cut Elimination** (and Proof Collapsing Techniques)
- Development of ever stronger **Ordinal Representation Systems**

- A **sequent** is an expression $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sequences of formulae $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$, respectively.

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- $\Gamma \Rightarrow \Delta$ is read, informally, as Γ yields Δ or, rather, the conjunction of the A_i yields the disjunction of the B_j.

Negation

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ \neg\mathrm{L} \qquad \frac{B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg B}\ \neg\mathrm{R}$$

Implication

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Lambda \Rightarrow \Theta}{A \to B, \Gamma, \Lambda \Rightarrow \Delta, \Theta}\ \to\mathrm{L} \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ \to\mathrm{R}$$

Conjunction

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \wedge L1 \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \wedge L2$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \wedge R$$

Disjunction

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \vee L$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \vee R1 \qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B} \vee R2$$

Quantifiers

$$\frac{F(t), \Gamma \Rightarrow \Delta}{\forall x\, F(x), \Gamma \Rightarrow \Delta}\ \forall L \qquad \frac{\Gamma \Rightarrow \Delta, F(a)}{\Gamma \Rightarrow \Delta, \forall x\, F(x)}\ \forall R$$

$$\frac{F(a), \Gamma \Rightarrow \Delta}{\exists x\, F(x), \Gamma \Rightarrow \Delta}\ \exists L \qquad \frac{\Gamma \Rightarrow \Delta, F(t)}{\Gamma \Rightarrow \Delta, \exists x\, F(x)}\ \exists R$$

In $\forall$L and $\exists$R, t is an arbitrary term. The variable a in $\forall$R and $\exists$L is an eigenvariable of the respective inference, i.e. a is not to occur in the lower sequent.

Identity Axiom

$$A \Rightarrow A$$

where A is any formula.

One could limit this axiom to the case of atomic formulae A

CUT

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Lambda \Rightarrow \Theta}{\Gamma, \Lambda \Rightarrow \Delta, \Theta}\text{ Cut}$$

A is called the cut formula of the inference.

Example

$$\frac{B \Rightarrow A \qquad A \Rightarrow C}{B \Rightarrow C}\text{ Cut}$$

Structural Rules **Exchange, Weakening, Contraction**

$$\frac{\Gamma, A, B, \Lambda \Rightarrow \Delta}{\Gamma, B, A, \Lambda \Rightarrow \Delta}\ \mathcal{X}_l \qquad \frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda}\ \mathcal{X}_r$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathcal{W}_l \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ \mathcal{W}_r$$

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathcal{C}_l \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathcal{C}_r$$

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Specifically, in the intuitionistic sequent calculus there are no inferences corresponding to contraction right or exchange right.

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Notice that the above proof is not intuitionistic since it involves sequents that are not intuitionistic.

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$$

Gentzen's Hauptsatz (1934)

Cut Elimination

If a sequent

$$\Gamma \Rightarrow \Delta$$

is provable, then it is provable without cuts.

Cut Elimination

EXAMPLE

Here is an example of how to eliminate cuts of a special form:

$$\dfrac{\dfrac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \to\mathrm{R} \qquad \dfrac{\Lambda \Rightarrow \Theta, A \qquad B, \Xi \Rightarrow \Phi}{A \to B, \Lambda, \Xi \Rightarrow \Theta, \Phi} \to\mathrm{L}}{\Gamma, \Lambda, \Xi \Rightarrow \Delta, \Theta, \Phi} \mathrm{Cut}$$

is replaced by

$$\dfrac{\dfrac{\Lambda \Rightarrow \Theta, A \qquad A, \Gamma \Rightarrow \Delta, B}{\Lambda, \Gamma \Rightarrow \Theta, \Delta, B} \mathrm{Cut} \qquad B, \Xi \Rightarrow \Phi}{\Gamma, \Lambda, \Xi \Rightarrow \Delta, \Theta, \Phi} \mathrm{Cut}$$

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Corollary *A contradiction, i.e. the empty sequent, is not deducible.*

Applications of the Haupsatz

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- Herbrand's Theorem in *LK* (classical):

$$\vdash \exists x R(x) \quad \text{implies} \quad \vdash R(t_1) \vee \ldots \vee R(t_n)$$

some t_i (R quantifier-free).

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- In **LJ** (intuitionistic predicate logic):

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  for some term t where Γ is $\vee$ and $\exists$ free.

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  for some term t where Γ is $\vee$ and $\exists$ free.
- Hilbert-Ackermann Consistency
- If T is a **geometric theory** and T classically proves a **geometric implication** A then T intuitionistically proves A.

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- What happens when we try to apply the procedure of cut elimination to theories?
- Axioms are detrimental to this procedure. It breaks down because the symmetry of the sequent calculus is lost. In general, we cannot remove cuts from deductions in a theory T when the cut formula is an axiom of T.
- However, sometimes the axioms of a theory are of bounded syntactic complexity. Then the procedure applies partially in that one can remove all cuts that exceed the complexity of the axioms of T.

Partial Cut Elimination

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- This is a very important tool in proof theory. For example, it works very well if the axioms of a theory can be presented as atomic intuitionistic sequents (also called Horn clauses), yielding the completeness of Robinsons **resolution method**.

Partial cut elimination also pays off in the case of fragments of **PA** and set theory with restricted induction schemes, be it induction on natural numbers or sets. This method can be used to extract bounds from proofs of Π^0_2 statements in such fragments.

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$$\mathrm{ord}(D) < \varepsilon_0.$$
- He then defined a reduction procedure $\mathcal{R}$ such that whenever D is a derivation of the empty sequent in **PA** then $\mathcal{R}(D)$ is another derivation of the empty sequent in **PA** but with a smaller ordinal assigned to it, i.e.,
$$\mathrm{ord}(\mathcal{R}(D)) < \mathrm{ord}(D). \tag{2}$$

Gentzen's way out

- Gentzen defined an assignment ord of ordinals to derivations of **PA** such for every derivation D of **PA** in his sequent calculus,

$$\mathrm{ord}(D) < \varepsilon_0.$$

- He then defined a reduction procedure $\mathcal{R}$ such that whenever D is a derivation of the empty sequent in **PA** then $\mathcal{R}(D)$ is another derivation of the empty sequent in **PA** but with a smaller ordinal assigned to it, i.e.,

$$\mathrm{ord}(\mathcal{R}(D)) < \mathrm{ord}(D). \qquad (2)$$

- Moreover, both ord and $\mathcal{R}$ are primitive recursive functions and only finitist means are used in showing (2).

Gentzen's way out cont'ed

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(i) *The theory of primitive recursive arithmetic,* **PRA**, *proves that* $\mathrm{PRWO}(\varepsilon_0)$ *implies the 1-consistency of* **PA**.

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Theorem: (Goodstein 1944, almost)

Termination of primitive recursive Goodstein sequences is not provable in **PA**.

Birth of Second Order Proof Theory by The Fundamental Conjecture on **GLC**

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> *Having proposed the fundamental conjecture, I concentrated on its proof and spent several years in an anguished struggle trying to resolve the problem day and night.*

The Finite Order Sequent Calculus, **GLC**

Quantifiers

$$\frac{F(\{v \mid A(v)\}), \Gamma \Rightarrow \Delta}{\forall X\, F(X), \Gamma \Rightarrow \Delta}\ \forall_2\,\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, F(U)}{\Gamma \Rightarrow \Delta, \forall X\, F(X)}\ \forall_2\,\mathrm{R}$$

$$\frac{F(U), \Gamma \Rightarrow \Delta}{\exists X\, F(X), \Gamma \Rightarrow \Delta}\ \exists_2\,\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, F(\{v \mid A(v)\})}{\Gamma \Rightarrow \Delta, \exists X\, F(X)}\ \exists_2\,\mathrm{R}$$

In $\forall_2$L and $\exists_2$R, $A(a)$ is an arbitrary formula. The variable U in $\forall_2$R and $\exists_2$L is an eigenvariable of the respective inference, i.e. U is not to occur in the lower sequent.

Non-constructive proofs of **FC**

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- G. Takeuti: *Consistency proofs of subsystems of classical analysis*, Ann. Math. 86 (1967) 299–348.

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A brief history of ordinal representation systems

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Hardy gives explicit representations for all ordinals $< \omega^2$.

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Veblen extended the initial segment of the countable for which fundamental sequences can be given effectively.

- He applied two new operations to **continuous increasing functions** on ordinals:
  - **Derivation**
  - **Transfinite Iteration**
- Let **ON** be the class of ordinals. A (class) function $f : \mathbf{ON} \to \mathbf{ON}$ is said to be **increasing** if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$ and **continuous** (in the order topology on **ON**) if

$$f(\lim_{\xi<\lambda} \alpha_\xi) = \lim_{\xi<\lambda} f(\alpha_\xi)$$

holds for every limit ordinal λ and increasing sequence $(\alpha_\xi)_{\xi<\lambda}$.

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- The **derivative** f' of a function f : **ON** $\to$ **ON** is the function which enumerates in increasing order the solutions of the equation

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- The **derivative** f' of a function $f : \mathbf{ON} \to \mathbf{ON}$ is the function which enumerates in increasing order the solutions of the equation

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also called the **fixed points** of f.
- If f is a normal function,

$$\{\alpha : f(\alpha) = \alpha\}$$

is a proper class and f' will be a normal function, too.

A Hierarchy of Ordinal Functions

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A Hierarchy of Ordinal Functions

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- $f_0 \;=\; f$
- $f_{\alpha+1} \;=\; {f_\alpha}'$
- $$f_\lambda(\xi) \;=\; \xi^{th} \text{ element of } \bigcap_{\alpha<\lambda} \{\text{Fixed points of } f_\alpha\} \qquad \text{for } \lambda \text{ limit.}$$

- From the normal function f we get a two-place function,

$$\varphi_f(\alpha, \beta) := f_\alpha(\beta).$$

Veblen then discusses the hierarchy when

$$f = \ell, \qquad \ell(\alpha) = \omega^\alpha.$$

The Feferman-Schütte Ordinal Γ_0

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$$\varphi_f(\alpha, \beta) := f_\alpha(\beta).$$

Veblen then discusses the hierarchy when

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- The least ordinal $\gamma > 0$ closed under φ_ℓ, i.e. the least ordinal > 0 satisfying

$$(\forall \alpha, \beta < \gamma)\, \varphi_\ell(\alpha, \beta) < \gamma$$

is the famous ordinal Γ_0 which Feferman and Schütte determined to be the least ordinal 'unreachable' by **predicative means**.

The Big Veblen Number

- Veblen extended this idea first to arbitrary **finite numbers of arguments**, but then also to **transfinite numbers of arguments**, with the proviso that in, for example

$$\Phi_f(\alpha_0, \alpha_1, \ldots, \alpha_\eta),$$

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only a finite number of the arguments

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may be non-zero.
- Veblen singled out the ordinal $E(0)$, where $E(0)$ is the least ordinal $\delta > 0$ which cannot be named in terms of functions

$$\Phi_\ell(\alpha_0, \alpha_1, \ldots, \alpha_\eta)$$

with $\eta < \delta$, and each $\alpha_\gamma < \delta$.

The Big Leap: H. Bachmann 1950

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- Define a set of ordinals $\mathfrak{B}$ closed under successor such that with each limit $\lambda \in \mathfrak{B}$ is associated an increasing sequence $\langle \lambda[\xi] : \xi < \tau_\lambda \rangle$ of ordinals $\lambda[\xi] \in \mathfrak{B}$ of length $\tau_\lambda \leq \mathfrak{B}$ and $\lim_{\xi<\tau_\lambda} \lambda[\xi] = \lambda$.

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- Let Ω be the first uncountable ordinal. A hierarchy of functions $(\varphi^{\mathfrak{B}}_\alpha)_{\alpha \in \mathfrak{B}}$ is then obtained as follows:

$$\varphi^{\mathfrak{B}}_0(\beta) = 1 + \beta \qquad \varphi^{\mathfrak{B}}_{\alpha+1} = \left(\varphi^{\mathfrak{B}}_\alpha\right)'$$

$$\varphi^{\mathfrak{B}}_\lambda \text{ enumerates } \bigcap_{\xi < \tau_\lambda} (\text{Range of } \varphi^{\mathfrak{B}}_{\lambda[\xi]}) \qquad \lambda \text{ limit, } \tau_\lambda < \Omega$$

$$\varphi^{\mathfrak{B}}_\lambda \text{ enumerates } \{\beta < \Omega : \varphi^{\mathfrak{B}}_{\lambda[\beta]}(0) = \beta\} \qquad \lambda \text{ limit, } \tau_\lambda = \Omega.$$

1960-1974

After Bachmann, the story of ordinal representation systems becomes very complicated.

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- **Feferman**'s new proposal: Bachmann-type hierarchy without fundamental sequences.

- **Bridge** and **Buchholz** showed computability of systems obtained by Feferman's approach.

"Natural" well-orderings

Set-theoretical (*Cantor, Veblen, Gentzen, Bachmann, Schütte, Feferman, Pfeiffer, Isles, Bridge, Buchholz, Pohlers, Jäger, Rathjen*)

- Define hierarchies of functions on the ordinals.
- Build up terms from function symbols for those functions.
- The ordering on the values of terms induces an ordering on the terms.

Reductions in proof figures (*Takeuti, Yasugi, Kino, Arai*)

- Ordinal diagrams; formal terms endowed with an inductively defined ordering on them.

"Natural" well-orderings

Patterns of elementary substructurehood (*Carlson*)

- Finite structures with Σ_n-elementary substructure relations .

Category-theoretical (*Aczel, Girard, Jervell, Vauzeilles*)

- Functors on the category of ordinals (with strictly increasing functions) respecting direct limits and pull-backs.

Representation systems from below (*Setzer*)

Second order arithmetic; $\mathbf{Z_2}$ *aka Analysis*

$\mathbf{Z_2}$ is a two sorted formal system. Extends **PA**.

- Variables $n, m, \ldots$ range over natural numbers.
  Variables $X, Y, Z, \ldots$ range over sets of natural numbers.
  Relation symbols $=, <, \in$. Function symbols $+, \times, \ldots$

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- Comprehension Principle/Axiom:

  For any property P definable in the language of $\mathbf{Z_2}$,

$$\{n \in \mathbb{N} \mid P(n)\}$$

  is a set; or more formally

$$\text{(CA)} \qquad \exists X \forall n [n \in X \leftrightarrow A(x)]$$

  for any formula $A(x)$ of $\mathbf{Z_2}$.

Stratification of Comprehension

- A Π^1_k-formula (Σ^1_k-formula) is a formula of $\mathbf{Z_2}$ of the form

  $$\forall X_1 \ldots QX_k\, A(X_1, \ldots, X_k) \qquad (\exists X_1 \ldots QX_k\, A(X_1, \ldots, X_k))$$

  with $\forall X_1 \ldots QX_k$ ($\exists X_1 \ldots QX_k$) a string of k alternating set quantifiers, beginning with a universal quantifier (existential quantifier), followed by a formula $A(X_1, \ldots, X_k)$ without set quantifiers.

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- Π^1_k-comprehension (Σ^1_k-comprehension) is the scheme

  $$\exists X\, \forall n\, [n \in X \leftrightarrow A(x)]$$

  with $A(x)$ Π^1_k (Σ^1_k).

Subsystems of $\mathbf{Z_2}$

- Basic arithmetical axioms in all subtheories of $\mathbf{Z_2}$ are: defining axioms for $0, 1, +, \times, E, <$ (as for **PA**) and the *induction axiom*

$$\forall X\,[0 \in X \wedge \forall n(n \in X \to n+1 \in X) \to \forall n\,(n \in X)].$$

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- For each axiom scheme **Ax**, $(\mathbf{Ax})_0$ denotes the theory consisting of the basic arithmetical axioms plus the scheme **Ax**.
- $(\mathbf{Ax})$ stands for the theory $(\mathbf{Ax})_0$ augmented by the scheme of induction for all $\mathcal{L}_2$-formulae.
- Let $\mathcal{F}$ be a collection of formulae of $\mathbf{Z_2}$.
  Another important axiom scheme for formulae F in $\mathcal{C}$ is

$$\mathcal{C}-\mathbf{AC} \qquad \forall n \exists Y F(n, Y) \to \exists Y \forall n F(x, Y_n),$$

where $Y_n := \{m : 2^n 3^m \in Y\}$.

How much of $\mathbf{Z_2}$ *is needed?*

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- Minimal foundational frameworks for Ordinary Mathematics:
  Feferman, **Lorenzen**, **Takeuti** ....
- **Reverse Mathematics**, early 1970s-now
  H. Friedman, **S. Simpson**, ...

  Given a specific theorem τ of ordinary mathematics, which set existence axioms are needed in order to prove τ?

Five Systems

For many mathematical theorems τ, there is a weakest natural subsystem $S(\tau)$ of $\mathbf{Z_2}$ such that $S(\tau)$ proves τ.
Moreover, it has turned out that $S(\tau)$ often belongs to a small list of specific subsystems of $\mathbf{Z_2}$. Reverse Mathematics has singled out five subsystems of $\mathbf{Z_2}$:

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- $(\Pi^1_1-\mathbf{CA})_0$ Π^1_1-Comprehension

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- **ATR_0** “Every countable reduced abelian *p*-group has an Ulm resolution"
- $(\Pi^1_1-\mathbf{CA})_0$ “Every uncountable closed set of real numbers is the union of a perfect set and a countable set"; “Every countable abelian group is a direct sum of a divisible group and a reduced group"

$|\mathbf{ATR}_0| = \Gamma_0$

$|\mathbf{ACA}_0| = \varepsilon_0$

$|\mathbf{RCA}_0| = \omega^\omega = |\mathbf{WKL}_0|$

0

$$|(\Sigma^1_2\text{-}\mathbf{AC}) + \mathbf{BI}| = \psi_{\Omega_1} I$$

$$|(\Delta^1_2\text{-}\mathbf{CA})| = \psi_{\Omega_1} \Omega_{\varepsilon_0}$$

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- Arai Ordinal Analysis of Theories up to Π^1_2-Comprehension using Reductions on Finite Proof Figures and Ordinal Diagrams.

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- IV. Combinatorial Independence Results

Examples

- I. (R; Setzer) Consistency proof of $(\Sigma^1_2\text{-}\mathbf{AC}) + \mathbf{BI}$ in Martin-Löf Type Theory.

Combinatorial Independence Results

- A *finite tree* is a finite partially ordered set

$$\mathbb{B} = (B, \leq)$$

such that:

(i) B has a smallest element (called the *root* of $\mathbb{B}$);

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- For finite trees $\mathbb{B}_1$ and $\mathbb{B}_2$, an embedding of $\mathbb{B}_1$ into $\mathbb{B}_2$ is a one-to-one mapping

$$f : \mathbb{B}_1 \to \mathbb{B}_2$$

  such that

$$f(a \wedge b) = f(a) \wedge f(b)$$

  for all $a, b \in \mathbb{B}_1$, where $a \wedge b$ denotes the infimum of a and b.

- **Kruskal's Theorem.** For every infinite sequence of trees $(\mathbb{B}_k : k < \omega)$, there exist i and j such that $i < j < \omega$ and $\mathbb{B}_i$ is embeddable into $\mathbb{B}_j$.
  (In particular, there is no infinite set of pairwise nonembeddable trees.)

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- **Theorem** (H. Friedman, D. Schmidt) Kruskal's Theorem is not provable in $\mathbf{ATR}_0$.

- The proof utilizes that Kruskal's Theorem implies that Γ_0 is well-founded.

The Extended Kruskal Theorem

- For $n < \omega$, let $\mathcal{B}_n$ be the set of all finite trees with labels from n, i.e. $(\mathbb{B}, \ell) \in \mathcal{B}_n$ if $\mathbb{B}$ is a finite tree and

$$\ell : B \to \{0, \ldots, n-1\}.$$

The set $\mathcal{B}_n$ is quasiordered by putting $(\mathbb{B}_1, \ell_1) \leq (\mathbb{B}_2, \ell_2)$ if there exists an embedding

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$$f : \mathbb{B}_1 \to \mathbb{B}_2 \qquad \text{such that:}$$

- $\ell_1(b) = \ell_2(f(b))$ for each $b \in B_1$;
- if b is an immediate successor of $a \in \mathbb{B}_1$, then for each $c \in \mathbb{B}_2$ in the interval $f(a) < c < f(b)$,

$$\ell_2(c) \geq \ell_2(f(b)).$$

This condition is called a **gap condition**.

Theorem. (Friedman) For each $n < \omega$, $\mathcal{B}_n$ is a **well quasi ordering** (abbreviated WQO($\mathcal{B}_n$)), i.e. there is no infinite set of pairwise nonembeddable trees.

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Theorem. (Friedman) For each $n < \omega$, $\mathcal{B}_n$ is a **well quasi ordering** (abbreviated $\mathrm{WQO}(\mathcal{B}_n)$), i.e. there is no infinite set of pairwise nonembeddable trees.

Theorem $\forall n < \omega \ \mathrm{WQO}(\mathcal{B}_n)$ is not provable in $\Pi^1_1 - \mathbf{CA}_0$.

- The proof employs an ordinal representation system for the proof-theoretic ordinal of $\Pi^1_1 - \mathbf{CA}_0$.
  The ordinal is $\psi_{\Omega_1}(\Omega_\omega)$.

The Graph Minor Theorem

- $\mathbb{G}$, $\mathbb{H}$ graphs. If $\mathbb{H}$ is obtained from $\mathbb{G}$ by first deleting some vertices and edges, and then contracting some further edges, $\mathbb{H}$ is a **minor** of $\mathbb{G}$.

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- The proof of GMT uses the EKT.
- **Corollary.** (Vázsonyi's conjecture) If all the $\mathbb{G}_k$ are trivalent, then there exist $i < j$ so that $\mathbb{G}_i$ is embeddable into $\mathbb{G}_j$.
- **Corollary.** (Wagner's conjecture) For any 2-manifold M there are only finitely many graphs which are not embeddable in M and are minimal with this property.

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  - GMT is not provable in $\Pi^1_1 - \mathbf{CA}_0$.

Ockham's Razor

In what follows, we shall be solely dealing with classical logic. Therefore we can simplify the sequent calculus as follows:

- We get rid of the **structural rules** by using **sets** of formulae rather than **sequents** of formulae. This has the effect that **exchange** and **contraction** happen automatically:

  $$\{C_1, \ldots, C_r, A, A, D_1, \ldots, D_s\} \;=\; \{D_1, \ldots, D_s, A, C_1, \ldots, C_r\}$$

  We take care of **weakening** by adding all the formulae we may be interested in from the start; thus we have more liberal axioms:

  $$A, \Gamma \;\Rightarrow\; \Delta, A$$

- Using the **De Morgan laws** of classical logic we can push **negations** in front of **atomic** formulae. Also, in classical logic $\neg, \wedge, \vee$ forms a **complete** set of connectives. Thus we can simplify matters, by demanding that formulae are built up from **atomic** and **negated atomic formulae** (literals) by means of $\wedge, \vee, \forall, \exists$.
  Negating a formula A then becomes a defined operation:
  - $\neg\neg A := A$ if A is atomic;
  - $\neg(A \wedge B) = \neg A \vee \neg B$; $\neg(A \vee B) = \neg A \wedge \neg B$;
  - $\neg\forall x\, F(x) := \exists x\, \neg F(x)$; $\neg\exists x\, F(x) := \forall x\, \neg F(x)$.
- In classical logic we don't need the two sides of a sequent

$$A_1, \ldots, A_r \Rightarrow \Delta$$

since it can be re-written as

$$\Rightarrow \neg A_1, \ldots, \neg A_r, \Delta$$

In the Tait-style version of the **classical sequent calculus** $\Gamma, \Delta, \Lambda, \Theta, \ldots$ range over finite sets of formulae in **negation normal form.** Γ, Δ stands for $\Gamma \cup \Delta$ and Δ, A is short for $\Delta \cup \{A\}$.

The **inferences** of the Tait-calculus are as follows:

(Axiom) $\Gamma, A, \neg A$

$$(\wedge) \quad \frac{\Gamma, A \qquad \Gamma, A'}{\Gamma, A \wedge A'}$$

$$(\vee) \quad \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1$$

$$(\forall) \quad \frac{\Gamma, F(a)}{\Gamma, \forall x\, F(x)}$$

$$(\exists) \quad \frac{\Gamma, F(t)}{\Gamma, \exists x\, F(x)}$$

$$(\text{Cut}) \quad \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

s an **eigenvariable**.

Part II: Predicative Proof Theory

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Theorem: (Schütte)

The proof-theoretic ordinal of $\mathbf{RA}_\alpha$ *is* $\varphi\alpha 0$.

Infinite Ramified Analysis $\mathbf{RA}^{\infty}$

Uses ordinal indexed free set variables $U^{\alpha}, V^{\alpha}, W^{\alpha}, \ldots$ and and bound set variables $X^{\beta}, Y^{\beta}, Z^{\beta}, \ldots$ with $\beta > 0$.

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3. If A and B are formulas of levels α and β, then $A \vee B$ and $A \wedge B$ are formulas of level $\max(\alpha, \beta)$.

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4. If $F(0)$ is a formula of level α, then $\forall x F(x)$ and $\exists x F(x)$ are formulas of level α and $\{x \mid F(x)\}$ is a **set term** of level α.

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5. If $F(U^\beta)$ is a formula of level α and $\beta > 0$, then $\forall X^\beta F(X^\beta)$ is a formula of level $\max(\alpha, \beta)$.

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4. $|\forall x B(x)| = |\exists x B(x)| = |t \in \{x \mid B(x)\}| = |t \notin \{x \mid B(x)\}| = |B(0)| + 1$

Cut rank in $\mathbf{RA}^{\infty}$

The **cut-rank** of a formula A, $|A|$, is defined as follows:

1. $|L| = 0$ for arithmetical literals L.
2. $|t \in U^{\alpha}| = |t \notin U^{\alpha}| = \omega \cdot \alpha$
3. $|B_0 \wedge B_1| = |B_0 \vee B_1| = \max(|B_0|, |B_1|) + 1$
4. $|\forall x B(x)| = |\exists x B(x)| = |t \in \{x \mid B(x)\}| = |t \notin \{x \mid B(x)\}| = |B(0)| + 1$
5. $|\forall X^{\alpha} A(X^{\alpha})| = |\exists X^{\alpha} A(X^{\alpha})| = \max(\omega \cdot \gamma, |A(U^0)| + 1)$

where γ is the level of $\forall X^{\alpha} A(X^{\alpha})$.

Cut-elimination for $\mathbf{RA}^\infty$

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Theorem:

Cut-elimination I:

$$\textit{If} \quad \mathbf{RA}^\infty \mathrel{\frac{|^{\alpha}}{|_{\eta+1}}} \Gamma \quad \textit{then} \quad \mathbf{RA}^\infty \mathrel{\frac{|^{\omega^\alpha}}{|_{\eta}}} \Gamma$$

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Theorem:

Cut-elimination II:

$$\textit{If} \quad \mathbf{RA}^\infty \left|\frac{\alpha}{\omega^\rho}\right. \Gamma \quad \textit{then} \quad \mathbf{RA}^\infty \left|\frac{\varphi\rho\alpha}{0}\right. \Gamma$$

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- Π^1_1-$\mathbf{CA}_0$ is an impredicative theory.

Kripke-Platek Set Theory

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- Gödel's Constructible Hierarchy **L**:

$$\begin{aligned} \mathbf{L}_0 &= \emptyset, \\ \mathbf{L}_\lambda &= \bigcup\{\mathbf{L}_\beta : \beta < \lambda\}\ \lambda \text{ limit} \\ \mathbf{L}_{\beta+1} &= \{X : X \subseteq \mathbf{L}_\beta;\ X \text{ definable over } \langle \mathbf{L}_\beta, \in \rangle\}. \end{aligned}$$

The **axioms** of **KP** are:

Extensionality:	$a = b \to [F(a) \leftrightarrow F(b)]$
Foundation:	$\exists x G(x) \to \exists x[G(x) \wedge (\forall y \in x)\neg G(y)]$
Pair:	$\exists x\ (x = \{a, b\}).$
Union:	$\exists x\ (x = \bigcup a).$
Infinity:	$\exists x\ \big[x \neq \emptyset\ \wedge\ (\forall y \in x)(\exists z \in x)(y \in z)\big].$
Δ_0 Separation:	$\exists x\ \big(x = \{y \in a : F(y)\}\big)$
	$F(y)$ Δ_0-formula.
Δ_0 Collection:	$(\forall x \in a)\exists y G(x, y) \to \exists z(\forall x \in a)(\exists y \in z)G(x, y)$
	for all Δ_0–formulas G.

By a Δ_0 formula we mean a formula of set theory in which all the quantifiers appear restricted, that is have one of the forms $(\forall x \in b)$ or $(\exists x \in b)$.

A set theory corresponding to Σ^1_2-AC + **BI**

The language of **KPi** is an extension of that **KP** by means of a unary predicate symbol Ad.
KPi is a set theory which comprises Kripke-Platek set theory and in addition has an axiom which says that any set is contained in an admissible set

$$\forall x\, \exists y\, [x \in y \ \wedge\ \mathrm{Ad}(y)]$$

$$\forall z\, [\mathrm{Ad}(z) \rightarrow \mathrm{Tran}(z) \ \wedge\ B^z]$$

for any axiom B of **KP**. Thus the standard models of **KPi** in **L** are the segments $\mathbf{L}_\kappa$ with κ recursively inaccessible. The ordinal analysis for **KPi** used an EORS built from ordinal functions which had originally been defined with the help of a weakly inaccessible cardinal. In this subsection we expound on the development of this particular EORS with an eye towards the role of cardinals therein.

Ordinal functions based on a weakly inaccessible cardinal

$$\mathbf{I} \quad := \quad \text{“first weakly inaccessible cardinal”} \tag{3}$$

$$(\alpha \mapsto \Omega_\alpha)_{\alpha<\mathbf{I}} \tag{4}$$

is a function that enumerates the cardinals below **I**. Further let

$$\mathfrak{R}^{\mathbf{I}} \quad := \quad \{\mathbf{I}\} \cup \{\Omega_{\xi+1} : \xi < \mathbf{I}\}. \tag{5}$$

Variables κ, π will range over $\mathfrak{R}^{\mathbf{I}}$.

Definition:

An ordinal representation system for the analysis of **KPi** can be derived from the following functions and Skolem hulls of ordinals defined by recursion on α:

$$C^{\mathbf{I}}(\alpha,\beta) = \begin{cases} \textit{closure of } \beta \cup \{0, \mathbf{I}\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^{\xi}) \\ (\xi \mapsto \Omega_{\xi})_{\xi < \mathbf{I}} \\ (\xi\pi \longmapsto \psi^{\xi}(\pi))_{\xi < \alpha} \end{cases}$$

$$\psi^{\alpha}(\pi) \simeq \min\{\rho < \pi : \; C^{\mathbf{I}}(\alpha,\rho) \cap \pi = \rho \; \wedge \; \pi \in C^{\mathbf{I}}(\alpha,\rho)\}.$$

Note that if $\rho = \psi^\alpha(\pi)$, then $\psi^\alpha(\pi) < \pi$ and $[\rho, \pi) \cap C^{\mathbf{I}}(\alpha, \rho) = \emptyset$, thus the order-type of the ordinals below π which belong to the Skolem hull $C^{\mathbf{I}}(\alpha, \rho)$ is ρ . In more pictorial terms, ρ is the α^{th} *collapse* of π.

Lemma:

If $\pi \in C^{\mathsf{I}}(\alpha, \pi)$, then $\psi^\alpha(\pi)$ is defined; in particular $\psi^\alpha(\pi) < \pi$.

Proof: Note first that for a limit ordinal λ,

$$C^{\mathsf{I}}(\alpha, \lambda) = \bigcup_{\xi < \lambda} C^{\mathsf{I}}(\alpha, \xi)$$

since the right hand side is easily shown to be closed under the clauses that define $C^{\mathsf{I}}(\alpha, \lambda)$. Thus we can pick $\omega \leq \eta < \pi$ such that $\pi \in C^{\mathsf{I}}(\alpha, \eta)$. Now define

$$\begin{aligned} \eta_0 &= \sup C^{\mathsf{I}}(\alpha, \eta) \cap \pi \\ \eta_{n+1} &= \sup C^{\mathsf{I}}(\alpha, \eta_n) \cap \pi \\ \eta^* &= \sup_{n<\omega} \eta_n. \end{aligned} \tag{6}$$

Since the cardinality of $C^{\mathsf{I}}(\alpha, \eta)$ is the same as that of η and therefore less than π, the regularity of π implies that $\eta_0 < \pi$.

By repetition of this argument one obtains $\eta_n < \pi$, and consequently $\eta^* < \pi$. The definition of η^* then ensures

$$C^{\mathsf{I}}(\alpha, \eta^*) \cap \pi \;=\; \bigcup_n C^{\mathsf{I}}(\alpha, \eta_n) \cap \pi \;=\; \eta^* < \pi.$$

Therefore, $\psi^\alpha(\pi) < \pi$. □

Let $\varepsilon_{\mathbf{I}+1}$ be the least ordinal $\alpha > \mathbf{I}$ such that $\omega^\alpha = \alpha$. The next definition singles out a subset $\mathcal{T}(\mathbf{I})$ of $C^{\mathbf{I}}(\varepsilon_{\mathbf{I}+1}, 0)$ which gives rise to an ordinal representation system, i.e., there is an elementary ordinal representation system $\langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle$, so that

$$\langle \mathcal{T}(\mathbf{I}), <, \Re, \psi, \ldots \rangle \;\cong\; \langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle. \tag{7}$$

"…" is supposed to indicate that more structure carries over to the ordinal representation system.

Definition:

$\mathcal{T}(\mathbf{I})$ is defined inductively as follows:

1. $0, \mathbf{I} \in \mathcal{T}(\mathbf{I})$.
2. If $\alpha_1, \ldots, \alpha_n \in \mathcal{T}(\mathbf{I})$ and $\alpha_1 \geq \ldots \geq \alpha_n$, then $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \in \mathcal{T}(\mathbf{I})$.
3. If $\alpha \in \mathcal{T}(\mathbf{I})$, $0 < \alpha < \mathbf{I}$ and $\alpha < \Omega_\alpha$, then $\Omega_\alpha \in \mathcal{T}(\mathbf{I})$.
4. If $\alpha, \pi \in \mathcal{T}(\mathbf{I})$, $\pi \in C^{\mathbf{I}}(\alpha, \pi)$ and $\alpha \in C^{\mathbf{I}}(\alpha, \psi^\alpha(\pi))$, then $\psi^\alpha(\pi) \in \mathcal{T}(\mathbf{I})$.

The side conditions in (2) and (3) are easily explained by the desire to have unique representations in $\mathcal{T}(\mathbf{I})$. The requirement $\alpha \in C^{\mathbf{I}}(\alpha, \psi^\alpha(\pi))$ in (4) also serves the purpose of unique representations (and more) but is probably a bit harder to explain. The idea here is that from $\psi^\alpha(\pi)$ one should be able to retrieve the stage (namely α) where it was generated. This is reflected by $\alpha \in C^{\mathbf{I}}(\alpha, \psi^\alpha(\pi))$.

the definition of $\mathcal{T}(\mathbf{I})$ is deterministic, that is to say every ordinal in $\mathcal{T}(\mathbf{I})$ is generated by the inductive clauses of in exactly one way. As a result, every $\gamma \in \mathcal{T}(\mathbf{I})$ has a unique representation in terms of symbols for $0, \mathbf{I}$ and function symbols for $+, (\alpha \mapsto \Omega_\alpha), (\alpha, \pi \mapsto \psi^\alpha(\pi)$. Thus, by taking some primitive recursive (injective) coding function $\ulcorner \cdots \urcorner$ on finite sequences of natural numbers, we can code $\mathcal{T}(\mathbf{I})$ as a set of natural numbers as follows:

$$\ell(\alpha) = \begin{cases} \ulcorner 0, 0 \urcorner & \text{if } \alpha = 0 \\ \ulcorner 1, 0 \urcorner & \text{if } \alpha = \mathbf{I} \\ \ulcorner 2, \ell(\alpha_1), \cdots, \ell(\alpha_n) \urcorner & \text{if } \alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \\ \ulcorner 3, \ell(\beta) \urcorner & \text{if } \alpha = \Omega_\beta \\ \ulcorner 4, \ell(\beta), \ell(\pi) \urcorner & \text{if } \alpha = \psi^\beta(\pi), \end{cases}$$

We have seen that in the case of **PA** the addition of an infinitary rule enables us to regain cut elimination.

ω–rule:

$$\frac{\Gamma, A(\bar{n}) \text{ for all } n}{\Gamma, \forall x\, A(x)}.$$

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An ordinal analysis for **PA** is then attained as follows:

- Each **PA**–proof can be "unfolded" into a $\mathbf{PA}_\omega$–proof of the same sequent.
- Each such $\mathbf{PA}_\omega$–proof can be transformed into a cut–free $\mathbf{PA}_\omega$–proof of the same sequent of length $< \varepsilon_0$.

In order to obtain a similar result for set theories like **KPi**, we have to work a bit harder. Guided by the ordinal analysis of **PA**, we would like to invent an infinitary rule which, when added to **KPi**, enables us to eliminate cuts.

The first ordinal analysis of **KPi** was given by Jäger, Pohlers in 1982.

As opposed to the natural numbers, it is not clear how to bestow a canonical name to each element of the set–theoretic universe.

Here we will use Gödel's constructible universe L. The constructible universe is “made" from the ordinals. It is pretty obvious how to “name” sets in L once we have names for ordinals at our disposal.

Recall that L_α, the αth level of **Gödel's constructible hierarchy** L, is defined by

$$\begin{aligned} L_0 &= \emptyset, \\ L_\lambda &= \bigcup\{L_\beta : \beta < \lambda\} \ \lambda \text{ limit} \\ L_{\beta+1} &= \{X : X \subseteq L_\beta;\ X \text{ definable over } \langle L_\beta, \in\rangle\}. \end{aligned}$$

So any element of L of level α is definable from elements of L with levels $< \alpha$ and the parameter L_{α_0} if $\alpha = \alpha_0 + 1$.

- Henceforth $\mathbf{I}$ will be a name for a large ordinal or even the whole class of ordinals.

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- The problem of “naming” sets will be solved by building a formal constructible hierarchy using the ordinals $\leq \mathbf{I}$.

Definition The $RS_{\mathbf{l}}$–terms and their levels are generated as follows.

1. For each $\alpha \leq \mathbf{l}$,

$$\mathbb{L}_\alpha$$

is an $RS_{\mathbf{l}}$–term of level α.

2. The formal expression

$$\{x \in \mathbb{L}_\alpha : F(x, \vec{s})^{\mathbb{L}_\alpha}\}$$

is an $RS_{\mathbf{l}}$–term of level α if $F(a, \vec{b})$ is an $\mathcal{L}$–formula (whose free variables are among the indicated) and $\vec{s} \equiv s_1, \cdots, s_n$ are $RS_{\mathbf{l}}$–terms with levels $< \alpha$.

$F(x, \vec{s})^{\mathbb{L}_\alpha}$ results from $F(x, \vec{s})$ by restricting all unbounded quantifiers to $\mathbb{L}_\alpha$.

Let $\mathcal{T}$ be the collection of all $RS_{\mathbf{l}}$-terms.
For $t \in \mathcal{T}$, $|t|$ denotes the level of t, i.e. the maximum ordinal α such that $\mathbb{L}_{\alpha}$ occurs in t.

We denote by upper case Greek letters

$$\Gamma, \Delta, \Lambda, \ldots$$

finite sets of $RS_{\mathbf{l}}$–formulae. The intended meaning of

$$\Gamma = \{A_1, \cdots, A_n\}$$

is the disjunction

$$A_1 \vee \cdots \vee A_n$$

Γ, A stands for $\Gamma \cup \{A\}$ etc..

The rules of $RS_{\mathbf{I}}$ are:

$$(\wedge) \quad \frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'}$$

$$(\vee) \quad \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1$$

$$(b\forall) \quad \frac{\cdots \Gamma, s \in t \to F(s) \cdots (|\, s \,| < |\, t \,|)}{\Gamma, (\forall x \in t) F(x)}$$

$$(b\exists) \quad \frac{\Gamma, s \in t \wedge F(s)}{\Gamma, (\exists x \in t) F(x)} \quad \text{if } |\, s \,| < |\, t \,|$$

$$(\notin) \qquad \frac{\cdots \Gamma, s \in t \to r \neq s \cdots\cdots (|s| < |t|)}{\Gamma, r \notin t}$$

$$(\in) \qquad \frac{\Gamma, s \in t \ \wedge \ r = s}{\Gamma, r \in t} \quad \text{if } |s| < |t|$$

$$(\text{Cut}) \qquad \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

$$(\text{Ref}_{\Sigma}(\pi)) \qquad \frac{\Gamma, A^{\mathbb{L}_\pi}}{\Gamma, (\exists z \in \mathbb{L}_\pi)\, A^z} \quad \text{if } A \text{ is a } \Sigma\text{-formula,}$$

where a formula is said to be in Σ if all its unbounded quantifiers are existential.

A^z results from A by restricting all unbounded quantifiers to z.

$\mathcal{H}$–controlled derivations

If we dropped the rules $(\text{Ref}_{\Sigma}(\pi))$ from $RS_{\mathbf{I}}$, the remaining calculus would enjoy full cut elimination owing to the symmetry of the pairs of rules

$$\begin{array}{cc} (\wedge) & (\vee) \\ (\forall) & (\exists) \\ (\notin) & (\in) \end{array}$$

However, partial cut elimination for $RS_{\mathbf{I}}$ can be attained by delimiting a collection of derivations of a very uniform kind. Buchholz developed a very elegant and flexible setting for describing uniformity in infinitary proofs, called operator controlled derivations.

Definition Let

$$P(ON) = \{X : X \text{ is a set of ordinals}\}.$$

A class function

$$\mathcal{H} : P(ON) \to P(ON)$$

will be called operator if $\mathcal{H}$ is a closure operator, i.e monotone, inclusive and idempotent, and satisfies the following conditions for all $X \in P(ON)$:

1. $0 \in \mathcal{H}(X)$.
2. If α has Cantor normal form $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, then $\alpha \in \mathcal{H}(X) \iff \alpha_1, ..., \alpha_n \in \mathcal{H}(X)$.

The latter ensures that $\mathcal{H}(X)$ will be closed under $+$ and $\sigma \mapsto \omega^\sigma$, and decomposition of its members into additive and multiplicative components.

For a term s, the operator $\mathcal{H}[s]$ is defined by

$$\mathcal{H}[s](X) \quad = \quad \mathcal{H}(X \cup \{\text{ all ordinals in } s\})$$

Definition Let $\mathcal{H}$ be an operator and let Γ be a finite set of $RS_{\mathbf{I}}$–formulae.

$$\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma$$

is defined by recursion on α. It is always demanded that

$$\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{H}(\emptyset).$$

The inductive clauses are:

$$(b\exists)\quad \frac{\mathcal{H} \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma, F(s)}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, (\exists x \in t)F(x)} \quad \begin{array}{r} \alpha_0 < \alpha \\ |s| < \alpha \\ |s| < |t| \end{array}$$

$$(b\forall)\quad \frac{\mathcal{H}[s] \,\big|\!\frac{\alpha_s}{\rho}\, \Gamma, F(s) \text{ for all } |s| < |t|}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, (\forall x \in t)F(x)} \quad |s| \leq \alpha_s < \alpha$$

$$(\mathrm{Cut})\quad \frac{\mathcal{H} \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma, B \qquad \mathcal{H} \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma, \neg B}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma} \quad \begin{array}{r} \alpha_0 < \alpha \\ rk(B) < \rho \end{array}$$

$$(Ref_\Sigma(\pi))\quad \frac{\mathcal{H} \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma, A^{\mathbb{L}_\pi}}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, (\exists z \in \mathbb{L}_\pi)\, A^z} \quad \begin{array}{r} \alpha_0, \Omega < \alpha \\ A \in \Sigma \end{array}$$

To connect **KPi** with the infinitary system $\mathit{RS}_{\mathbf{I}}$ one has to show that **KPi** can be embedded into $\mathit{RS}_{\mathbf{I}}$. Indeed, the finite **KPi**-derivations give rise to very uniform infinitary derivations.

Theorem:

If

$$\mathbf{KPi} \vdash B(a_1, \ldots, a_r)$$

then

$$\mathcal{H} \mathrel{\Big|\frac{\mathsf{I}\cdot m}{\mathsf{I}+n}} B(s_1, \ldots, s_r)$$

holds for some m, n *and all set terms* $s_1, \ldots, s_r$ *and operators* $\mathcal{H}$ *satisfying*

$$\{\xi \,:\, \xi \text{ occurs in } B(\vec{s})\} \subseteq \mathcal{H}(\emptyset).$$

m and n depend only on the **KPi***-derivation of* $B(\vec{a})$.

Das Ende

OBRIGADO!