Model theory (analytic part)

Mário Edmundo
U. Aberta & CMAF/UL

Days in Logic 2014

The tutorial

The tutorial

- A bit of o-minimality

The tutorial

- A bit of o-minimality
- A bit of o-minimality and Gronthendieck

The tutorial

- A bit of o-minimality
- A bit of o-minimality and Gronthendieck
- A bit of o-minimality and André-Oort.

A bit of o-minimality and Andre-Oort

A bit of o-minimality and Andre-Oort

Applications: a special case

Applications: a special case

...

Applications: a special case

...

Theorem (Manin-Mumford conjecture)

Let A be an abelian variety and X an algebraic sub variety of A, both defined over a number field. Suppose that X does no contain any translate of an abelian sub variety of A of dimension > 0. Then X contains only finitely many torsion points of A.

Applications: a special case

...

Theorem (Manin-Mumford conjecture)

Let A be an abelian variety and X an algebraic sub variety of A, both defined over a number field. Suppose that X does no contain any translate of an abelian sub variety of A of dimension > 0. Then X contains only finitely many torsion points of A.

Proved by Raynaud (1983), later other proofs all using methods from arithmetic geometry...

Applications: a special case

...

Theorem (Manin-Mumford conjecture)

Let A be an abelian variety and X an algebraic sub variety of A, both defined over a number field. Suppose that X does no contain any translate of an abelian sub variety of A of dimension > 0. Then X contains only finitely many torsion points of A.

Proved by Raynaud (1983), later other proofs all using methods from arithmetic geometry...

Pila and Zanier (2008) give a new proof <u>using o-minimality</u> which they <u>generalize to other cases</u> of André-Oort type conjectures...

Applications: a special case

Applications: a special case

Special case of abelian variety A are the elliptic curves, given by equations of the form:

$$y = x^3 + ax + b$$

with $a, b \in \mathbb{Q}$ (see picture....)

Applications: a special case

Special case of abelian variety A are the elliptic curves, given by equations of the form:

$$y = x^3 + ax + b$$

with $a, b \in \mathbb{Q}$ (see picture....)

They:

- have a group operation $(A, \oplus)$ also given (on charts) by rational functions $\frac{P(\overline{x})}{Q(\overline{x})}$ where $P(\overline{x}), Q(\overline{x})$ are polynomials with coefficients in some number field (i.e. finite extension of $\mathbb{Q}$).
- are complete (like “compact”)

Applications: a special case

Applications: a special case

- an abelian sub variety is another abelian variety B which is a subgroup.
- an T-torsion point is a point $x \in A$ such that $\underbrace{x \oplus \ldots \oplus x}_{\text{T times}} = e$ (the identity).
- a translate is something of the form $x \oplus B$.
- an algebraic variety is something defined on charts by

$$\{\overline{x} \in \mathbb{C}^n : P_1(\overline{x}) = \ldots = P_k(\overline{x}) = 0\}$$

where $P_i(\overline{x})$ are polynomials with coefficients in some number field (i.e. finite extension of $\mathbb{Q}$).

Applications: a special case

Applications: a special case

.... ingredients from arithmetic geometry

Theorem (Masser (1984))

Suppose A is defined over a number field K. If x is a T-torsion point of A, then

$$[K(x) : \mathbb{Q}] \geq c_2(A) T^{\rho}$$

for $c_2(A) > 0$ and $\rho > 0$ which depend only on $\dim A$. In particular,

$$\# \text{ conjugates of } x \text{ is } \geq c_3(A) T^{\rho}.$$

Applications: a special case

Applications: a special case

.... ingredients from o-minimality

Applications: a special case

.... ingredients from o-minimality

Let

$$\mathcal{M} = (\mathbb{R}, 0, 1, -, +, \cdot, (f)_{f \in \mathcal{F}}, (R)_{R \in \mathcal{R}}, <)$$

be an o-minimal structure over the field of real numbers.

Applications: a special case

.... ingredients from o-minimality

Let

$$\mathcal{M} = (\mathbb{R}, 0, 1, -, +, \cdot, (f)_{f \in \mathcal{F}}, (R)_{R \in \mathcal{R}}, <)$$

be an o-minimal structure over the field of real numbers.

If Z is definable, let

$$Z^{\mathrm{alg}}$$

be the union of all definably connected, <u>semi-algebraic subsets</u> of Z of positive dimension.

Applications: a special case

.... ingredients from o-minimality

Let

$$\mathcal{M} = (\mathbb{R}, 0, 1, -, +, \cdot, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}}, <)$$

be an o-minimal structure over the field of real numbers.

If Z is definable, let

$$Z^{\mathrm{alg}}$$

be the union of all definably connected, semi-algebraic subsets of Z of positive dimension.

If $T \in \mathbb{N}$ and $W \subseteq \mathbb{R}^n$ is a set, let

$$N(W, T) = \#\{\overline{q} \in W \cap \mathbb{Q}^n : \text{ denominators of } \overline{q} \text{ divide } T\}$$

Applications: a special case

Applications: a special case

....

Theorem (Pila and Wilkie (2006))

Let Z be a definable set. For every $\epsilon > 0$ we have

$$N(Z \setminus Z^{\mathrm{alg}}, T) \leq c_1(Z, \epsilon) T^{\epsilon}.$$

Applications: a special case

....

Theorem (Pila and Wilkie (2006))

Let Z be a definable set. For every $\epsilon > 0$ we have

$$N(Z \setminus Z^{\mathrm{alg}}, T) \leq c_1(Z, \epsilon) T^{\epsilon}.$$

This follows from a parametrization (after Gromov) result for definable sets and functions in o-minimal expansions of real closed field... this is like a dual of C^r-Cell decomposition theorem.

Applications: a special case

Applications: a special case

.... the new proof of Manin-Mumford:

Applications: a special case

.... the new proof of Manin-Mumford:

There is a <u>complex analytic uniformization</u>

$$0 \to \Lambda \to \mathbb{C}^g \to A \to 0$$

periodic with period lattice Λ, where $g = \dim A$.

Applications: a special case

.... the new proof of Manin-Mumford:

There is a complex analytic uniformization

$$0 \to \Lambda \to \mathbb{C}^g \to A \to 0$$

periodic with period lattice Λ, where $g = \dim A$.

- $\mathbb{C}^g/\Lambda$ is a complex abelian Lie group, so a torus.
- sub-torus of $\mathbb{C}^g/\Lambda$ corresponde to abelian sub-varieties of A (by Chow's theorem).

Applications: a special case

Applications: a special case

....

Applications: a special case

....

Fix a $\mathbb{Z}$-basis $\lambda_1, \ldots, \lambda_{2g}$ of Λ and use it to identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$.

Applications: a special case

....

Fix a $\mathbb{Z}$-basis $\lambda_1, \ldots, \lambda_{2g}$ of Λ and use it to identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$.

- T-torsions of A correspond to $\overline{q} \in \mathbb{Q}^{2g}$ with denominators dividing T.

Theorem (Peterzil-Starchenko)

There is a fundamental domain $H \subseteq \mathbb{C}^g$ such that the restriction of the uniformization $\mathbb{C}^g \to A$ is definable in

$$\overline{\mathbb{R}}_{\mathrm{an}} = (\mathbb{R}, 0, 1, -, +, \cdot, (f)_{f \in an}, <).$$

- a subvariety X of A corresponds to a definable set $Z \subseteq H$.

Applications: a special case

Applications: a special case

....

Applications: a special case

....

Lemma

If X a sub variety of A does not contain a translate of an abelian sub variety of A of positive dimension, then $Z^{\mathrm{alg}} = \emptyset$.

...this is where an actual proof is needed in the paper... it uses also finiteness properties of o-minimal structures.

Applications: a special case

....

Lemma

If X a sub variety of A does not contain a translate of an abelian sub variety of A of positive dimension, then $Z^{\mathrm{alg}} = \emptyset$.

...this is where an actual proof is needed in the paper... it uses also finiteness properties of o-minimal structutures.

Conclusion of proof:

- by Masser $c_3(A)T^{\rho} \leq N(Z, T)$.
- by Pila-Wilkie with $\epsilon = \frac{\rho}{2}$ we get

$$c_3(A)T^{\rho} \leq N(Z, T) \leq c_1(A, \rho)T^{\frac{\rho}{2}}$$

- $\#T$-torsions in X is bounded as $T \to +\infty$, so is finite $\square$

Applications: André-Oort

Applications: André-Oort

...

Applications: André-Oort

...

Conjecture (André-Ort type conjectures)

Let A be an algebraic variety of suitable type and X an algebraic sub variety of A, both defined over a number field. Suppose that X does no contain any special sub variety of A of dimension > 0. Then X contains only finitely many special points of A.

Applications: André-Oort

Applications: André-Oort

...

Applications: André-Oort

...

Proved by Klingler, Ullmo and Yafaev (2007), under GRH.

Applications: André-Oort

...

Proved by Klingler, Ullmo and Yafaev (2007), under GRH.

Pila extends the new proof of Manin-Mumford to other special cases which were unknown unconditionally.

Applications: André-Oort

Applications: André-Oort

...in one of these other special cases:

Applications: André-Oort

...in one of these other special cases:

- $A = \mathbb{C}^2$ parametrizing pairs of elliptic curves, up to isomorphism, by their j-invariant.
- $(j, j') \in A = \mathbb{C}^2$ is special, if j, j' are both j-invariants of CM elliptic curves.
- the special sub varieties of positive dimension are:
  - $\{z, z_2) \in \mathbb{C}^2 : z \text{ is a CM } j\text{-invariant}\}$
  - $\{z_1, z) \in \mathbb{C}^2 : z \text{ is a CM } j\text{-invariant}\}$
  - modular curves defined by $F_N(z, z') = 0$ where for each N, F_N is such that $F_N(j(\tau), j(N\tau)) = 0$ for all $\tau \in \mathcal{H}$.
  - $\mathbb{C}^2$.

Applications: André-Oort

Applications: André-Oort

...the proof is similar with the following replacements:

Applications: André-Oort

...the proof is similar with the following replacements:

- lower bound on T-torsion by lower bounds on discriminates of CM fields;
- uniformization by action of Λ on $\mathbb{C}^g$ by uniformization by action of $\mathrm{SL}_2(\mathbb{Z})$ on upper half plane $\mathcal{H}$;
- $\overline{\mathbb{R}}_{\mathrm{an}}$ by $\overline{\mathbb{R}}_{\mathrm{an},exp}$.
- upper bound of $N(Z, T)$ by upper bound of $N_k(Z, T)$

THANK YOU!