Model theory (analytic part)

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Days in Logic 2014

The tutorial

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- A bit of o-minimality

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- A bit of o-minimality
- A bit of o-minimality and Gronthendieck

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- A bit of o-minimality
- A bit of o-minimality and Gronthendieck
- A bit of o-minimality and André-Oort.

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O-minimal structures

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- semi-algebraic geometry: definable sets in real closed fields
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- sub-analytic geometry: definable sets in the field of real numbers expanded by restricted analytic functions
$$\overline{\mathbb{R}}_{\mathrm{an}} = (\mathbb{R}, 0, 1, -, +, \cdot, (f)_{f \in \mathrm{an}}, <).$$

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Do they:

- capture tameness?
- provides new insights originated from model-theoretic methods into the real analytic-like setting?

About tameness

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van den Dries (1984), Knight, Pillay and Steinhorn (1986):

Theorem (Cell decomposition)

- (I_n) Let $A_1, \ldots, A_k \subseteq M^n$ be definable. Then exists a cell decomposition $\mathcal{D}$ of M^n compatible with the A_i's
- (II_n) Let $f : A \subseteq M^n \to M$ be definable. Then exists a cell decomposition $\mathcal{D}$ of M^n compatible with A such that for each $D \in \mathcal{D}$ we have $f_{|D} : D \to M$ is continuous.

About tameness

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The proof of cell decomposition is by induction on n. Assuming (I_n) and (II_n) we first get (III_n) below. From (I_n), (II_n) and (III_n) we get (I_{n+1}) and (II_{n+1}).

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Lemma (Uniform finiteness property)

(III_n) Let $A \subseteq M^{n+1}$ be definable such that for all $\overline{x} \in M^n$ the fiber $A_{\overline{x}} = \{t \in M : (\overline{x}, t) \in A\}$ is finite. Then exists N_A such that $\#A_{\overline{x}} \leq N_A$ for all $\overline{x} \in M^n$.

About tameness

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Theorem (Monotonicity theorem)
Let $f : (a, b) \subseteq M \to M$ be definable. Then exists

$$a_0 = a < a_1 < \ldots < a_k < a_{k+1} = b$$

such that each $f_| : (a_i, a_{i+1}) \to M$ is either constant, or strictly monotone a continuous.

About tameness

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Sketch of proof of (I_{n+1}):

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Let:

$Y = \bigcup_{i=1}^{k}\{(x,t) \in M^{n+1} : t \in \mathrm{bd}(A_{i\,x})\}$ and take N with $\# Y_x \leq N$ ... by (III_n).

$B_l = \{x \in M^n : \# Y_x = l\}$ and take $f_{lj} : B_l \to Y$ with $(Y_{|B_l})_x = \{f_{l1}(x), \ldots, f_{ll}(x)\}$ and $-\infty = f_{l0} < f_{l1} < \ldots < f_{ll} < f_{ll+1} = +\infty$.

$C_{ilj} = \{x \in B_l : f_{lj}(x) \in (A_i)_x\}$ and $D_{ilj} = \{x \in B_l : (f_{lj}(x), f_{lj+1}(x)) \subseteq (A_i)_x\}$.

About tameness

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....:

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Apply (I_n) and (II_n) to B_l's, C_{ilj}'s, D_{ilj}'s and the f_{lj}'s. Let $\mathcal{D}$ the cell decomposition. Take

$$\mathcal{D}^* = \bigcup \{\mathcal{D}_E : E \in \mathcal{D}\}$$

where for each $E \subseteq B_l$

$$\mathcal{D}_E = \{(f_{lj|E}, f_{lj+1|E})'s, \Gamma(f_{lj|E})'s\}.$$

Then $\mathcal{D}^*$ is a cell decomposition of M^{n+1} which partitions each $A_1, \ldots, A_k$. □

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So let $f : A \subseteq M^{n+1} \to M$ be definable. By (I_{n+1}) we may assume that A is a cell.

Case (1): A is a cell and non open in M^{n+1}.

By construction of cells, exists $p : A \to p(A) \subseteq M^k$ with $k \leq n$, a projection which is a definable homeomorphism, such that $p(A)$ is an open cell in M^k. To finish apply (II_k) to $f \circ p^{-1} : p(A) \to M$.

About tameness

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Case (2): A is an open cell in M^{n+1}.

Let A^* be the definable subset of A of all (z, t) such that exists open box $C \times (a, b) \subseteq A$ such that:

(a) $z \in C$;

(b) $\forall x \in C$, $f(x, -) : (a, b) \to M$ is continuous and monotone;

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Fix some open box $C \times (a,c) \subseteq A$. Let $\lambda : C \to (a,c)$ be such that $\lambda(x) = \max\{s \in (a,c] : f(x,-) : (a,s) \to M$ is continuous and monotone $\}$. By Monotonicity theorem λ is well defined and definable. By (II_n) we assume λ is continuous. Fix $b \in (a,c)$ and taking again a smaller C we may assume $b \leq \lambda(x)$ for all $x \in C$. Fix $t \in (a,b)$, by (II_n) we assume $f(-,t) : C \to M$ is continuous. So $C \times (a,b) \cap A^* \neq \emptyset$ and A^* is dense in A.

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By (I_{n+1}) let $\mathcal{D}$ be a cell decomposition of M^{n+1} compatible with A^* and A. It is enough to show that $f_{|D} : D \to M$ is continuous for $D \in \mathcal{D}$ open cell such that $D \subseteq A$.

But then $D \subseteq A^*$, so for all $(z, t) \in D$ such that exists open box $C \times (a, b) \subseteq D$ such that:

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By easy general topology $f_| : C \times (a,c) \to M$ is continuous on each such open box. So $f_{|D} : D \to M$ is continuous. $\square$

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Corollary (Łojasiewicz property)

Let $A \subseteq M^n$ be definable in $\mathcal{M}$. Then A has finitely many definably connected components.

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Corollary (Łojasiewicz property)

Let $A \subseteq M^n$ be definable in $\mathcal{M}$. Then A has finitely many definably connected components.

Corollary (Uniform Łojasiewicz property)

Let $A \subseteq M^m \times M^n$ be definable in $\mathcal{M}$. Then there is $N_A \in \mathbb{N}$ such that for each $x \in M^m$, the fiber $A_x \subseteq M^n$ has at most N_A many definably connected components.

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... we have a notion of <u>dimension</u> of definable sets $A \subseteq M^n$:

$$\dim A = \max\{\dim C : C \subseteq A \text{ a cell}\}$$

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Theorem

For definable sets we have:

- If $A \subseteq B$ then $\dim A \leq \dim B$;
- $\dim(B \cup C) = \max\{\dim B, \dim C\}$;
- If $S \subseteq M^{m+n}$ then each

$$S(d) = \{x \in M^m : \dim\ S_x = d\}$$

is <u>definable</u> and

$$\dim\Big(\bigcup_{x \in S(d)} \{x\} \times S_x\Big) = \dim(S(d)) + d.$$

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Theorem

Let S be non empty definable set. Then

$$\dim \partial S < \dim S.$$

In particular, $\dim \mathrm{cl}(S) = \dim S$.

About tameness

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... a stratification $\mathfrak{G}$ of a closed definable set $A \subseteq M^n$ is a partition of A into finitely many cells, called strata of $\mathfrak{G}$, such that for each stratum $C \in \mathfrak{G}$ its frontier ∂C is a union of lower dimension strata.

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Theorem (Existence of stratifications)

Let $A \subseteq M^n$ be non empty closed definable set and $A_1, \ldots, A_k$ definable subsets of A. Then exists a stratification of A partitioning each of $A_1, \ldots, A_k$.

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is an o-minimal structure on a real closed field $(R, 0, 1, -, +, \cdot, <)$...

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- C^k-stratifications for any fixed k;
- Definable triangulation theorem;
- Definable trivialization theorem;
- $\ldots\ldots$

(the others depending on your motivation...)

About new insights

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There are many important o-minimal expansions

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of the ordered field of real numbers

$$\overline{\mathbb{R}},\ \overline{\mathbb{R}}_{\mathrm{an}},\ \overline{\mathbb{R}}_{\mathrm{exp}},\ \overline{\mathbb{R}}_{\mathrm{an,\,exp}},\ \overline{\mathbb{R}}_{\mathrm{an}^*},\ \overline{\mathbb{R}}_{\mathrm{an}^*,\,\mathrm{exp}},\ \overline{\mathbb{R}}_{\mathrm{Pfaff}},\ \overline{\mathbb{R}}_{QA}, \ldots$$

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constructed to include the exponential function, restoration of Riemmann zeta function, restriction of gamma function, Rolle leaves, classes of C^∞ quasi-analytic functions,...

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constructed to include the exponential function, restoration of Riemmann zeta function, restriction of gamma function, Rolle leaves, classes of C^∞ quasi-analytic functions,...

In each of these new structures our <u>tameness results</u> of course apply... which was <u>not known</u> before.

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Bierstone and Milman:

“An understanding of the behaviour at infinity of certain important classes of sub-analytic sets as in Wilkie’s (1996)

$$\overline{\mathbb{R}}_{\exp} = (\mathbb{R}, 0, 1, +, \cdot, \exp, <)$$

represents the most striking success of the model-theoretic point of view in sub-analytic geometry.”

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In particular we have tameness in the following non-standard o-minimal models of resp. semi-algebraic and sub-analytic geometry:

- $\mathbb{R}((t^{\mathbb{Q}})) = (\mathbb{R}((t^{\mathbb{Q}})), 0, 1, +, \cdot, <)$
- $\mathbb{R}((t^{\mathbb{Q}}))_{\mathrm{an}} = (\mathbb{R}((t^{\mathbb{Q}})), 0, 1, +, \cdot, (f)_{f \in \mathrm{an}}, <)$

Grothendieck again

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We would like to develop a theory of sheaves on definable spaces in arbitrary o-minimal structures

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generalizing/in analogy to:

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- the theory of sheaves in semi-algebraic geometry (Delfs);
- the theory of sheaves in algebraic geometry (Grothendieck);
- the theory of sheaves on locally compact topological spaces (Verdier).

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... we have to use sites (Grothendieck topologies), the o-minimal site X_{def}.

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This is very deep and has applications to the theory of D-modules.

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Theorem (Delfs)

The natural morphism of sites

$$\mu : \mathrm{Spec}_r R[V] \longrightarrow V_{\mathrm{sa}}$$

induces an isomorphism

$$\mathrm{Mod}(k_{V_{\mathrm{sa}}}) \longrightarrow \mathrm{Mod}(k_{\mathrm{Spec}_r R[V]})$$

of the corresponding categories of sheaves of k-modules.

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For a real analytic manifold X consider the natural morphism

$$\rho : X \longrightarrow X_{sa}$$

of sites and the induced functors

$$\mathrm{Mod}^{c}_{\mathbb{R}-c}(k_X) \subset \mathrm{Mod}(k_X) \underset{\rho^{-1}}{\overset{\rho_*}{\rightleftarrows}} \mathrm{Mod}(k_{X_{sa}}).$$

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The restriction of ρ_* extends to an equivalence of categories

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Moreover, $F \simeq \varinjlim_{i} \rho_* F_i, \ \{F_i\}_{i \in I} \in \mathrm{Mod}^{c}_{\mathbb{R}-c}(k_X)$.

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The natural morphism of sites

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... never used in sub-analytic case ... connects logic to real algebra.

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Comparing the two methods

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With the second method:

- can transfer classical results only if X is locally compact;
- the category $\mathrm{Ind}(\bullet)$ is complicated.

Results: o-minimal cohomology

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We can develop o-minimal sheaf cohomology by defining as usual

$$H^q(X;F) := H^q(\widetilde{X};\widetilde{F}) = R^q\Gamma(\widetilde{X};\widetilde{F})$$

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Theorems (E, Peatfield and Jones)

- Vanishing Theorem.
- Vietoris-Begle Theorem.
- Eilenberg-Steenrod Axioms.

Results: o-minimal local Verdier duality

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Theorem (E, Prelli)

There exists $\mathcal{D}^*$ in $\mathrm{D}^+(k_{X_{\mathrm{def}}})$ and a natural isomorphism

$$\mathrm{RHom}_{k_{X_{\mathrm{def}}}}(\mathcal{F}^*, \mathcal{D}^*) \simeq \mathrm{RHom}_k(R\Gamma_c(X, \mathcal{F}^*), k)$$

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... conjectured by Delf's in the semi-algebraic case.

Results: o-minimal Poincaré and Alexander duality

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Theorems (E, Prelli)

Let X be definable manifold of dimension n.

- If X has an orientation k-sheaf $\mathcal{O}r_X$, then

$$H^p(X; \mathcal{O}r_X) \simeq H_c^{n-p}(X; \underline{k})^\vee.$$

- If X is k-orientable and Z is a closed definable subset, then

$$H_Z^p(X; k_X) \simeq H_c^{n-p}(Z; \underline{k})^\vee.$$

Results: sub-analytic case

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Kashiwara-Schapira (resp. L. Prelli) define the operators

$$Rf_*, \;\; f^{-1}, \;\; \otimes^L, \;\; R\mathcal{H}om, \;\; Rf_{!!}, \;\; f^!$$

by setting

$$f_{!!}\, \text{“}\varinjlim_i\text{”} F_i := \text{“}\varinjlim_i\text{”} f_! F_i$$

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.... and develop the formalism of these six Grothendieck operations.

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The formalism of the six Grothendieck operations

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- Base Change Theorem:

$$g^{-1}Rf_{!!}\mathcal{F} \simeq Rf'_{!!}g'^{-1}\mathcal{F}.$$

- Projection Formula:

$$Rf_{!!}\mathcal{F} \otimes \mathcal{G} \simeq Rf_{!!}(\mathcal{F} \otimes f^{-1}\mathcal{G}).$$

- Künneth Formula:

$$R\delta_{!!}(g'^{-1}\mathcal{F} \otimes f'^{-1}\mathcal{G}) \simeq Rf_{!!}\mathcal{F} \otimes Rg_{!!}\mathcal{G}.$$

- Global form of Verdier duality:

$$\mathrm{Hom}(\mathcal{F}, f^{!}\mathcal{G}) \simeq \mathrm{Hom}(Rf_{!!}\mathcal{F}, \mathcal{G}).$$

O-minimal six operations

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In recent work with L. Prelli we define the operators

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by setting, in the tilde world:

$$\Gamma(U; f_{\wr!}F) := \varinjlim_{Z} \Gamma_Z(f^{-1}(U); F)$$

with Z closed constructible subsets of $f^{-1}(U)$ such that $f_{|Z} : Z \longrightarrow U$ is proper (i.e., separated and universally closed).

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THANK YOU!