Gaps in the Milnor-Moore spectral sequence

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Abstract

The Toomer invariant of a simply connected space X, $\mathrm{e}(X)$, is the least integer k for which the inclusion $B_k\Omega X \hookrightarrow B\Omega X$, where $B_k\Omega X$ is the kth stage of the classifying construction on ΩX, is surjective in homology. The Toomer invariant of X is a lower bound of the Lusternik-Schnirelmann category of X. We construct CW-complexes Z and $Z \cup e^m$ such that $\mathrm{e}(Z) = 2$ and $\mathrm{e}(Z \cup e^m) = 4$. This exhibits the Toomer invariant as the first approximation of the L.-S. category which fails to increase by at most one when a cell is attached to a space. We deduce from our result that there may be gaps in the Milnor-Moore spectral sequence in the sense that one may have $E^\infty_{p,*} = 0$ and $E^\infty_{p+1,*} \neq 0$.

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Introduction

The Lusternik-Schnirelmann category of a space X, denoted $\mathrm{cat}(X)$, is the least integer n such that X can be covered by $n+1$ open sets each of which is contractible in X. If no such n exists one sets $\mathrm{cat}(X) = \infty$. Originally the L.-S. category was introduced as an invariant which gave a lower bound for the number of critical points of a differentiable function defined on a manifold. In spite of the simplicity of its definition category is not an easy homotopy invariant, neither from a theoretical nor from a computational point of view. Numerous approximations, such as the weak categories of I. Berstein and P. J. Hilton [3] and of W. Gilbert [7], the rational category [4], and the A-category of S. Halperin and J.-M. Lemaire [9], have been introduced in order to handle the difficulties with the category.

Let $\mathbf{k}$ be a commutative ring. In 1963 M. Ginsburg [8] showed that for a simply connected space X $\mathrm{cat}(X)$ is always greater than or equal to the greatest integer p for which the Milnor-Moore spectral sequence of X satisfies $E^\infty_{p,*} \neq 0$. Recall that the Milnor-Moore spectral sequence of a simply connected space X converges to $H_*(X;\mathbf{k})$ and that $E^2 = \mathrm{Tor}^{H_*(\Omega X)}(\mathbf{k},\mathbf{k})$ when $\mathbf{k}$ is a field. The number $\sup\{p \in \mathbb{N}, E^\infty_{p,*} \neq 0\}$ has been studied by G. H. Toomer [15] who showed, in particular, that it coincides with the least integer n for which the morphism $H_*(B_n\Omega X;\mathbf{k}) \to H_*(B\Omega X;\mathbf{k})$, induced by the inclusion of the nth stage of the classifying construction on ΩX into the classifying space $B\Omega X$, is surjective. Today this number is known as the *Toomer invariant* of X and is denoted by $\mathrm{e}_{\mathbf{k}}(X)$.

In order to attain a better understanding of the category and its approximations it is natural to compare these homotopy invariants, and besides the numerical aspect of the question it is interesting to compare the invariants through their properties. For example, the category increases by at most one when a cell is attached to a space and so do, for instance, the rational category, the weak category of Berstein-Hilton [3], and the A-category [10]. In the course of a workshop in Oberwolfach in 1997, Y. Félix asked whether this property is also shared by the rational Toomer invariant. In this work we answer in the negative to this question. We construct CW-complexes Z and $Z \cup e^m$ satisfying $\mathrm{e}_{\mathbb{Q}}(Z) = 2$ and $\mathrm{e}_{\mathbb{Q}}(Z \cup e^m) = 4$. As a

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consequence we obtain that there may be gaps in the Milnor-Moore spectral sequence in the sense that one may have $E^{\infty}_{p,*} = 0$ and $E^{\infty}_{p+1,*} \neq 0$. Notice that such a phenomenon cannot occur at the level of the E^2 term of the Milnor-Moore spectral sequence. Indeed, if $\mathbf{k}$ is a field and X is a simply connected space, then $\mathrm{Tor}_p^{H_*(\Omega X)}(\mathbf{k}, \mathbf{k}) = 0$ implies $\mathrm{Tor}_{p+1}^{H_*(\Omega X)}(\mathbf{k}, \mathbf{k}) = 0$.

Through our result the rational Toomer invariant appears as the first example of an approximation of the category which fails to increase by at most one when a cell is attached to a space. By computing them for the spaces Z and $Z \cup e^m$ we show that the integral Toomer invariant $\mathrm{e}_{\mathbb{Z}}$, the weak category of Gilbert, the strict category weight of Yu. B. Rudyak [12], and the invariants $\sigma^i\mathrm{cat}$ defined in [16] fail also to have this property.

Acknowledgment. We are indebted to Yves Félix for a very useful conversation which enabled us to improve our results significantly. The space $Z \cup e^m$ had originally 76 cells! Now it has 6 cells and its rational cohomology is a Poincaré duality algebra.

1 Preliminaries

Throughout this article a *space* is a well-pointed compactly generated Hausdorff space of the homotopy type of a CW-complex. Any continuous map preserves the base point. A CW-complex is *1-reduced* if it has no 1-cells and only one 0-cell. A space is *of finite type* if it has the homotopy type of a CW-complex having finitely many cells in each dimension. For a space X we denote by ΩX the Moore loop space of X.

We fix a commutative ring $\mathbf{k}$. (Graded) modules are always (graded) $\mathbf{k}$-modules. All homology groups are to be taken with coefficients in $\mathbf{k}$. For a space X we denote by $C_*(X)$ the normalized singular chain complex of X with coefficients in $\mathbf{k}$.

If V is a chain complex, then the homology class of a cycle $z \in V$ will be denoted by $\{z\}$. The suspension of a graded module $V = (V)_{n\in\mathbb{Z}}$ is the graded module sV defined by $(sV)_n = V_{n-1}$. We follow the convention $V^n = V_{-n}$. (Differential) graded algebras are assumed to be associative and augmented. (Differential) graded coalgebras are assumed to be coassociative and coaugmented. For a graded (co)algebra B we denote by $\bar{B}$ the (co)augmentation (co)ideal of B. For a graded module V we denote by TV the tensor (co)algebra on V. When TV is the tensor coalgebra on V we shall write for elements $v_1, \ldots, v_n \in V$ $[v_1|\cdots|v_n]$ instead of $v_1 \otimes \cdots \otimes v_n$. A differential graded (co)algebra B is *connected* if it is non negatively graded and $B_0 = \mathbf{k}$. A *chain algebra* is a differential graded algebra A such that $A_n = 0$ for $n < 0$. A *cochain algebra* is a differential graded algebra A such that $A^n = 0$ for $n < 0$.

A morphism of differential graded modules (algebras, coalgebras) is called a *weak equivalence* if it induces an isomorphism in homology. A morphism of filtered differential graded modules (algebras, coalgebras) is a weak equivalence if it is a homology isomorphism at each level of the filtration. Weak equivalences are denoted by the symbol $\xrightarrow{\simeq}$. Two objects V and W of a category with weak equivalences are said to be *weakly equivalent* if they are connected by a finite sequence of weak equivalences $V \xrightarrow{\simeq} \cdot \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} \cdot \xleftarrow{\simeq} W$. We consider the homotopy equivalences as the weak equivalences in the category of topological spaces. Two spaces are thus weakly equivalent if and only if they are homotopy equivalent. If a category has weak equivalences, so does the morphism category. We can thus speak of weakly equivalent continuous maps, chain maps, differential graded algebra morphisms etc.

The multiplication of the Moore loop space ΩX induces a multiplication on the chain complex $C_*(\Omega X)$ which turns the latter into a chain algebra. Adams and Hilton have shown that for any simply connected space X there is a chain algebra (TV, d) with V positively graded and $\mathbf{k}$-free which is weakly equivalent to $C_*(\Omega X)$. Any such chain algebra will be called an *Adams-Hilton model* of X.

Definition 1.1. The *bar construction* on a differential graded algebra A is the differential graded coalgebra $BA = (T(s\bar{A}), d_1 + d_2)$ where d_1 and d_2 are given by

$$d_1[sa_1|\ldots|sa_n] = -\sum_{i=1}^{n}(-1)^{\varepsilon(i)}[sa_1|\ldots|sda_i|\ldots|sa_n],$$

$$d_2[sa_1|\ldots|sa_n] = \sum_{i=2}^{n}(-1)^{\varepsilon(i)}[sa_1|\ldots|sa_{i-1}a_i|\ldots|sa_n].$$

Here, $\varepsilon(1) = 0$ and $\varepsilon(i) = i - 1 + \sum_{j=1}^{i-1} |a_j|$ for $i > 1$.

The bar construction is a functor in the obvious way. It is well known that the bar construction turns weak equivalences between differential graded algebras which are free as $\mathbf{k}$-modules into weak equivalences.

Consider a differential graded algebra (TV, d) and form the natural d-stable graded submodule

$$sT^{>1}(V) \oplus T^{>1}(s\overline{TV})$$

of $B(TV, d)$. The quotient of $B(TV, d)$ by this sub chain complex is the graded module $\mathbf{k} \oplus sV$ with a differential $\bar{d}$. We denote by p the canonical projection $B(TV, d) \to (\mathbf{k} \oplus sV, \bar{d})$.

Proposition 1.2. [5, 19.1] *If V is positively graded, then the projection $p : B(TV, d) \to (\mathbf{k} \oplus sV, \bar{d})$ is a weak equivalence.* □

2 L.-S. category and the Milnor-Moore spectral sequence

Recall from the introduction that the Lusternik-Schnirelmann category of a space X, denoted $\mathrm{cat}(X)$, is the least integer n such that X can be covered by $n + 1$ open sets each of which is contractible in X. If no such n exists one sets $\mathrm{cat}(X) = \infty$. The L.-S. category can be characterized by means of the following sequence of fibrations due to T. Ganea: Let $g_0(X) : G_0(X) \to X$ be the path fibration $PX \to X$. Suppose that the nth fibration $g_n(X) : G_n(X) \to X$ has been defined. In order to define the $n + 1$st fibration take the fibre $F_n(X)$ of $g_n(X)$ and form the map $(g_n(X), *) : G_n(X) \cup_{F_n(X)} CF_n(X) \to X$. The fibration $g_{n+1}(X) : G_{n+1}(X) \to X$ is then defined to be the mapping path fibration associated to the map $(g_n(X), *)$. It is well known that the maps $g_n(X)$ can also be described as follows: Let ΩX denote the Moore loop space of X. The classifying space $B\Omega X$ of the topological monoid ΩX comes equipped with an increasing filtration of subspaces $B_0\Omega X \subset B_1\Omega X \subset \cdots \subset B_n\Omega X \subset \cdots$. If X is simply connected, the fibration $g_n(X)$ and the inclusion $B_n\Omega X \hookrightarrow B\Omega X$ are weakly equivalent. The link between the category and the Ganea fibrations is given by the following theorem:

Theorem 2.1. [6] *For a path-connected space X we have cat $X \leq n$ if and only if the fibration $g_n(X) : G_n(X) \to X$ has a section.* □

In this text we will mainly be concerned with the following approximation of the category:

Definition 2.2. The *Toomer invariant* of space X, denoted by $\mathrm{e}_{\mathbf{k}}(X)$, is the least integer n for which the morphism $H_*(g_n(X); \mathbf{k}) : H_*(G_n(X); \mathbf{k}) \to H_*(X; \mathbf{k})$ is surjective. If no such integer exists we set $\mathrm{e}_{\mathbf{k}}(X) = \infty$. We shall write $\mathrm{e}(X)$ and $\mathrm{e}_0(X)$ instead of $\mathrm{e}_{\mathbb{Z}}(X)$ and $\mathrm{e}_{\mathbb{Q}}(X)$.

It follows immediately from the definition that for a path-connected space X $\mathrm{e}_{\mathbf{k}}(X) \leq \mathrm{cat}(X)$. Thanks to the following theorem by M. Ginsburg $\mathrm{e}_{\mathbf{k}}(X)$ can, if X is simply connected, be calculated from an Adams-Hilton model of X. For a differential graded algebra A we denote by B_nA the sub differential graded coalgebra $T^{\leq n}(s\bar{A})$ of BA.

Theorem 2.3. [8] *For a simply connected space X the filtered chain complexes $C_*(B_*\Omega X)$ and $B_*C_*(\Omega X)$ are weakly equivalent.* □

The spectral sequence associated with the filtered differential module $B_*C_*(\Omega X)$ is the *Milnor-Moore spectral sequence* of X. It is well known that the Milnor-Moore spectral sequence converges to $H_*(X)$ when X is simply connected and that $E^2_{p,q} = \mathrm{Tor}^{H_*(\Omega X)}_{p,q}(\mathbf{k}, \mathbf{k})$ when, furthermore, $\mathbf{k}$ is a field. Using 2.3, Toomer established the following theorem:

Theorem 2.4. [15] *Let X be a simply connected space of finite category and $\{E^r\}$ be the Milnor-Moore spectral sequence of X. Then $\mathrm{e}_{\mathbf{k}}(X) = sup\{p \in \mathbb{N} | E^{\infty}_{p,*} \neq 0\}$.* □

It follows from this that $\mathrm{cat}(X) \geq \sup\{p \in \mathbb{N} | E^{\infty}_{p,*} \neq 0\}$. This inequality appeared first in Ginsburg [8].

For the rest of this section we suppose that $\mathbf{k}$ *is a field.*

For a differential graded algebra (TV, d) we denote by p_n the restriction to $B_n(TV, d)$ of the canonical projection $p : B(TV, d) \to (\mathbf{k} \oplus sV, \bar{d})$. Our calculations of the Toomer invariant will be based on the following proposition:

Proposition 2.5. *Let X be a simply connected space and (TV, d) be an Adams-Hilton model of X. Then $\mathrm{e}_{\mathbf{k}}(X) \leq n$ if and only if the inclusion $B_n(TV, d) \hookrightarrow B(TV, d)$ is surjective in homology. If $n > 0$, then this is the case if and only if the projection $p_n : B_n(TV, d) \to (\mathbf{k} \oplus sV, \bar{d})$ has a differential section.*

Proof: Since (TV, d) and $C_*(\Omega X)$ are weakly equivalent chain algebras, the morphisms $B_n(TV, d) \hookrightarrow B(TV, d)$ and $B_n C_*(\Omega X) \hookrightarrow BC_*(\Omega X)$ are weakly equivalent. By 2.3, these morphisms are weakly equivalent to the chain map $C_*(B_n \Omega X) \hookrightarrow C_*(B\Omega X)$ and thus to the chain map $C_*(g_n(X)) : C_*(G_n(X)) \to C_*(X)$. It follows that $\mathrm{e}_{\mathbf{k}}(X) \leq n$ if and only if the inclusion $B_n(TV, d) \hookrightarrow B(TV, d)$ is surjective in homology. By 1.2, this is the case if and only if the projection $p_n : B_n(TV, d) \to (\mathbf{k} \oplus sV, \bar{d})$ is surjective in homology. For $n > 0$ the map p_n is surjective. Since, over a field, a surjective chain map has a section if and only if it is surjective in homology, it follows that for $n > 0$ we have $\mathrm{e}_{\mathbf{k}}(X) \leq n$ if and only if p_n has a section. □

The main objective of this article is to show the Toomer invariant fails to have the basic property of the category to increase by at most one when a cell is attached to a space. This is equivalent to the fact that gaps can occur in the Milnor-Moore spectral sequence in the sense that one may have $E^{\infty}_{p,*} = 0$ but $E^{\infty}_{p+1,*} \neq 0$. More precisely:

Proposition 2.6. *Let X be a simply connected space and $f : S^n \to X$ $(n > 0)$ be a map. If there exists an integer p such that $\mathrm{e}_{\mathbf{k}}(X) < p$ and $\mathrm{e}_{\mathbf{k}}(X \cup_f D^{n+1}) = p + 1$, then the Milnor-Moore spectral sequence of $X \cup_f D^{n+1}$ satisfies $E^{\infty}_{p,*} = 0$ and $E^{\infty}_{p+1,*} \neq 0$. Conversely, if the Milnor-Moore spectral sequence of a 1-reduced CW-complex Y, whose cellular chain complex differential is zero, satisfies $E^{\infty}_{r,*} = 0$ and $E^{\infty}_{r+1,*} \neq 0$ for some $r \in \mathbb{N}$, then Y contains subcomplexes Q and $Q \cup e^m$ such that $\mathrm{e}_{\mathbf{k}}(Q) + 1 < \mathrm{e}_{\mathbf{k}}(Q \cup e^m)$.*

Proof: Let first X be a simply connected space and $f : S^n \to X$ $(n > 0)$ be a map such that there exists an integer p for which $\mathrm{e}_{\mathbf{k}}(X) < p$ and $\mathrm{e}_{\mathbf{k}}(X \cup_f D^{n+1}) = p + 1$. Let $A = (TV, d)$ be an Adams-Hilton model of X. Thanks to [2] we may suppose that the differential $\bar{d}$ of $\mathbf{k} \oplus sV$ is zero. Adjoin a generator to A and extend the differential d to construct an Adams-Hilton model $U = (T(V \oplus \mathbf{k}e), d)$ of $X \cup_f D^{n+1}$. Then the differential $\bar{d}$ of $\mathbf{k} \oplus s(V \oplus \mathbf{k}e)$ is zero. Otherwise the inclusion $(\mathbf{k} \oplus sV, 0) \hookrightarrow (\mathbf{k} \oplus s(V \oplus \mathbf{k}e), \bar{d})$ would be surjective in homology and this would imply that $\mathrm{e}_{\mathbf{k}}(X \cup_f D^{n+1}) \leq \mathrm{e}_{\mathbf{k}}(X) < p$. We denote by F_k the image of the homomorphism $H_* B_k U \to H_* BU = \mathbf{k} \oplus s(V \oplus \mathbf{k}e)$. We then have $E^{\infty}_{k,l} = (F_k / F_{k-1})_{k+l}$. As $\mathrm{e}_{\mathbf{k}}(X \cup_f D^{n+1}) = p + 1$, we have $F_p \subsetneqq F_{p+1} = \mathbf{k} \oplus s(V \oplus \mathbf{k}e)$ and hence $E^{\infty}_{p+1,*} \neq 0$. As $\mathrm{e}_{\mathbf{k}}(X) < p$, the morphism $H_* B_k A \to H_* BA = \mathbf{k} \oplus sV$ is surjective for $k \geq p - 1$. Since the homomorphism $H_* BA \to H_* BU$ induced by the inclusion $BA \hookrightarrow BU$ is the inclusion $\mathbf{k} \oplus sV \hookrightarrow \mathbf{k} \oplus s(V \oplus \mathbf{k}e)$, it follows that $\mathbf{k} \oplus sV \subset F_k$ for $k \geq p - 1$. Since $F_p \subsetneqq \mathbf{k} \oplus s(V \oplus \mathbf{k}e)$, it follows that $F_{p-1} = F_p = \mathbf{k} \oplus sV$ and thus that $E^{\infty}_{p,*} = 0$.

Let now Y be a 1-reduced CW-complex such that the differential in the cellular chain complex of Y is zero and such that the Milnor-Moore spectral sequence of Y satisfies $E^{\infty}_{r,*} = 0$ and $E^{\infty}_{r+1,*} \neq 0$ for some $r \in \mathbb{N}$. Let $R = (TW, d)$ be an Adams-Hilton model of Y which is constructed as described in [1]. Then the cellular chain complex of Y is isomorphic to the differential graded module $(\mathbf{k} \oplus sW, \bar{d})$. By assumption, $\bar{d} = 0$. Let $\mathcal{B} \subset W$ be a basis the elements of which correspond to the cells of Y. As $E^{\infty}_{r+1,*} \neq 0$, there exists an element of $\mathcal{B}$ which is not in the image of the homomorphism $H_* B_r R \to H_* BR = \mathbf{k} \oplus sW$. Suppose that $x \in \mathcal{B}$ is such an element with minimal degree $m - 1$. Let Q be the $m - 1$ skeleton of Y and e^m be the cell corresponding to x. Then Q and $Q \cup e^m$ are subcomplexes of Y and the sub differential graded algebras $S = (T(W_{<m-1}), d)$ and $T = (T(W_{<m-1} \oplus \mathbf{k}x), d)$ of R are Adams-Hilton models of Q and $Q \cup e^m$. By the minimality of $m - 1$, we have $\mathbf{k} \oplus sW_{<m-1} \subset \mathrm{im}(H_* B_r R \to H_* BR)$. Since $E^{\infty}_{r,*} = 0$, it follows that $\mathbf{k} \oplus sW_{<m-1} \subset \mathrm{im}(H_* B_{r-1} R \to H_* BR)$. As the inclusion $B_{r-1} S \hookrightarrow B_{r-1} R$ is surjective in homology up to degree $m - 1$ and the map $H_* BS \to H_* BR$ induced by the inclusion $BS \hookrightarrow BR$ is the inclusion $\mathbf{k} \oplus sW_{<m-1} \hookrightarrow \mathbf{k} \oplus sW$, we obtain that the inclusion $B_{r-1} S \hookrightarrow BS$ is surjective in homology and hence

that $e_{\mathbf{k}}(Q) \leq r-1$. Since the map $H_*BT \to H_*BR$ induced by the inclusion $BT \hookrightarrow BR$ is the inclusion $\mathbf{k} \oplus s(W_{<m-1} \oplus \mathbf{k}x) \hookrightarrow \mathbf{k} \oplus sW$, the element x cannot be in the image of the morphism $H_*B_rT \to H_*BT$ as it otherwise would also lie in the image of the morphism $H_*B_rR \to H_*BR$. It follows that $e_{\mathbf{k}}(Q \cup e^m) > r$ and hence that $e_{\mathbf{k}}(Q) + 1 < e_{\mathbf{k}}(Q \cup e^m)$. □

At the E^2 term of the Milnor-Moore spectral sequence there cannot be any gaps as is showing the following well known result:

Proposition 2.7. *If X is a simply connected space, then $Tor_p^{H_*(\Omega X)}(\mathbf{k},\mathbf{k}) = 0$ implies $Tor_{p+1}^{H_*(\Omega X)}(\mathbf{k},\mathbf{k}) = 0$.*

Proof: Set $A = H_*(\Omega X)$ and consider the free resolution of the graded A-module $\mathbf{k}$

$$0 \leftarrow \mathbf{k} \overset{\varepsilon}{\leftarrow} A \otimes V_0 \overset{d_1}{\leftarrow} A \otimes V_1 \leftarrow \cdots \overset{d_p}{\leftarrow} A \otimes V_p \overset{d_{p+1}}{\leftarrow} A \otimes V_{p+1} \leftarrow \cdots$$

where $V_0 = \mathbf{k}$, ε is the augmentation, $V_{p+1} = s(\ker d_p / \bar{A} \ker d_p)$, and d_{p+1} is the A-linear extension of a section of the projection (of degree 1) $\ker d_p \to V_{p+1}$. Then $\operatorname{im} d_{p+1} \subset \bar{A} \otimes V_p$. It follows that the differential of the DG vector space V obtained from the resolution by killing the action of A is zero. This identifies $V_p = \mathrm{Tor}_p^A(\mathbf{k},\mathbf{k})$. By construction, if $V_p = 0$, then also $V_{p+1} = 0$. The result follows. □

3 The rational Toomer invariant

In this section we construct CW-complexes Z and $Z \cup e^{16}$ such that $e_0(Z) = 2$ and $e_0(Z \cup e^{16}) = 4$. We suppose that $\mathbf{k} = \mathbb{Q}$.

The space Z is the CW-complex $S^2 \vee S^3 \cup e^8 \cup e^{13} \cup e^{14}$ where the cells are attached as follows. The attaching map of e^8 is the composite $[S^2, S^3] \circ \eta$ where $\eta : S^7 \to S^4$ is the Hopf map. The cells e^{13} and e^{14} are attached by the Whitehead products $[S^2, \phi \circ \omega]$ and $[S^3, \phi \circ \omega]$ where $\omega : S^{11} \to \mathbb{H}P^2$ is the attaching map of the top cell of $\mathbb{H}P^3$ and ϕ is the cobase extension of the Whitehead product $[S^2, S^3] : S^4 \to S^2 \vee S^3$ by the inclusion $S^4 \hookrightarrow \mathbb{H}P^2$.

In order to define the attaching map $\gamma : S^{15} \to Z$ of the cell e^{16} in the CW-complex $Z \cup e^{16}$ we look at a Quillen model of Z. Recall that a Quillen model of a space X is a differential graded Lie algebra representing the rational homotopy type of X. References on Quillen models include [2], [14], and [5]. For the convenience of the reader we recall that a *graded Lie algebra* is a graded vector space L with a bracket $[\,,\,] : L \otimes L \to L$ satisfying $[x,y] = -(-1)^{|x||y|}[y,x]$ (antisymmetry) and $[[x,y],z] = [x,[y,z]] - (-1)^{|x||y|}[y,[x,z]]$ (Jacobi identity). A *differential graded Lie algebra* is a couple (L,d) consisting of a graded Lie algebra L and a boundary operator d satisfying $d([x,y]) = [dx,y] + (-1)^{|x|}[x,dy]$. For a graded vector space V we denote by $\mathbb{L}(V)$ the free graded Lie algebra on V. A Quillen model of Z is given by the differential graded Lie algebra $(\mathbb{L}(V), d)$ where V is the graded vector space generated by elements x, y, u, a, and b of degrees 1, 2, 7, 12, and 13 and the differential is given by

$$dx = 0, \qquad dy = 0, \qquad du = \frac{1}{2}[[x,y],[x,y]], \qquad da = [x,[[x,y],u]], \qquad \text{and} \qquad db = [y,[[x,y],u]].$$

The fact that the differential graded Lie algebra $(\mathbb{L}(V), d)$ represents the rational homotopy type of Z means in particular that there is an isomorphism (of degree 1) $\tau : H_*(\mathbb{L}(V), d) \xrightarrow{\cong} \pi_*(Z) \otimes \mathbb{Q}$ (cf. [5, 24(b)]). This isomorphism converts the Lie bracket in $H_*(\mathbb{L}(V), d)$ to the Whitehead product by the rule $\tau[\alpha,\beta] = (-1)^{|\alpha|}[\tau\alpha, \tau\beta]$ (cf. [5, 24.5]). The element $z = [x,b] + [y,a] + \frac{1}{2}[u,u]$ is a cycle in $(\mathbb{L}(V), d)$. Define the attaching map $\gamma : S^{15} \to Z$ of the cell e^{16} to be a cellular map such that $\tau\{mz\} = [\gamma] \otimes 1$ for some positive integer m.

A Quillen model of $Z \cup e^{16}$ is given by the differential graded Lie algebra $(\mathbb{L}(V \oplus \mathbb{Q}e), d)$ where d extends the differential on $\mathbb{L}(V)$ and $de = z$. We remark that, by the link between the quadratic part of the differential of a minimal Quillen model and the cup product in rational cohomology (cf. for ex. [2, 2.14]), $H^*(Z \cup e^{16})$ is a Poincaré duality algebra.

Recall that the *universal enveloping algebra* of a graded Lie algebra L is the graded algebra $UL = TL/I$ where I is the ideal generated by the elements $xy - (-1)^{|x||y|}yx - [x,y]$, $x, y \in L$. The universal enveloping algebra of a free Lie algebra $\mathbb{L}(W)$ is the tensor algebra TW. If (L,d) is a differential graded Lie algebra, then UL is canonically a differential graded algebra. Baues and Lemaire [2] have shown that if $(\mathbb{L}(W), d)$ is a Quillen model of a simply connected space of finite type X, then the differential graded algebra $(U\mathbb{L}(W), d) = (TW, d)$ is an Adams-Hilton model of X. We therefore have

Proposition 3.1. *The differential graded algebra $(U\mathbb{L}(V), d) = (TV, d)$ is an Adams-Hilton model of Z and the differential graded algebra $(U\mathbb{L}(V \oplus \mathbb{Q}e), d) = (T(V \oplus \mathbb{Q}e), d)$ is an Adams-Hilton model of $Z \cup e^{16}$.* □

The basis $\mathcal{B} = \{x, y, u, a, b, e\}$ of the graded vector space $V \oplus \mathbb{Q}e$ induces a basis $\mathcal{M}_n$ of the graded vector space $\bigoplus_{i=1}^{n} T^i(s\overline{T(V \oplus \mathbb{Q}e)}) = \overline{B_n T(V \oplus \mathbb{Q}e)}$. The elements of $\mathcal{M}_n$ will be called *monomials*. The element $[s(x^3ey^2x)|s(y^3u)|s(u^2)]$ is a typical monomial in $\mathcal{M}_n$, $n \geq 3$. In order to lighten the presentation we will suppress the s's from the notation and write $[x^3ey^2x|y^3u|u^2]$ instead of $[s(x^3ey^2x)|s(y^3u)|s(u^2)]$. We denote by $< , >$ the symmetric bilinear form on the graded vector space $\overline{B_n T(V \oplus \mathbb{Q}e)}$ defined on monomials by

$$< m, m' >= \begin{cases} 1 & m = m', \\ 0 & m \neq m'. \end{cases}$$

Theorem 3.2. *The spaces Z and $Z \cup e^{16}$ satisfy $\mathrm{e}_0(Z) = 2$ and $\mathrm{e}_0(Z \cup e^{16}) = 4$.*

Proof: We first show that $\mathrm{e}_0(Z) = 2$. From degree 2 on the projection $B_1(TV, d) \to (\mathbb{Q} \oplus sV, 0)$ is just the suspension of the projection $(TV, d) \to (V, 0)$. As the element $u \in V_7 = H_7(V, 0)$ is not in the image of the homomorphism $H_7(TV, d) \to H_7(V, 0)$ (which is null), the homomorphism $H_8 B_1(TV, d) \to H_8(\mathbb{Q} \oplus sV, 0)$ is not surjective. This shows that $\mathrm{e}_0(Z) \geq 2$. A section ι of the projection $B_2(TV, d) \to (\mathbb{Q} \oplus sV, 0)$ is given by $\iota(1) = 1$, $\iota(x) = x$, $\iota(y) = y$, $\iota(u) = u + [[x, y]|[x, y]]$, $\iota(a) = a + [x|[[x, y], u]] + [[[x, y], u]|x]$, $\iota(b) = b - [y|[[x, y], u]] + [[[x, y], u]|y]$. As the section ι commutes with the differentials, we have $\mathrm{e}_0(Z) = 2$.

We now show that $\mathrm{e}_0(Z \cup e^{16}) = 4$. We clearly have $\mathrm{cat}(Z \cup e^{16}) \leq 4$ and therefore $\mathrm{e}_0(Z \cup e^{16}) \leq 4$. It remains to show that $\mathrm{e}_0(Z \cup e^{16}) \geq 4$. Let n be an integer such that the projection

$$p_n : B_n(T(V \oplus \mathbb{Q}e), d) \to (\mathbb{Q} \oplus s(V \oplus \mathbb{Q}e), 0)$$

has a differential section σ. We show that $n \geq 4$. Set $\xi = e - \sigma e$. Clearly, $\xi \in \ker p_n$. As $\mathcal{M}_n \backslash \mathcal{B}$ is a basis of

$$\ker p_n = sT^{>1}(V \oplus \mathbb{Q}e) \oplus \bigoplus_{i=2}^{n} T^i(s\overline{T(V \oplus \mathbb{Q}e)}),$$

we have $\xi = \sum_{m \in \mathcal{M}_n \backslash \mathcal{B}} < \xi, m > m$. As σ is a differential section, we have

$$d\xi = de - d\sigma e = de - \sigma d_{\mathbb{Q} \oplus s(V \oplus \mathbb{Q}e)} e = de.$$

It follows that $< d\xi, u^2 >=< de, u^2 >\neq 0$. We thus have

$$0 \neq< d\xi, u^2 >=< d(\sum_{m \in \mathcal{M}_n \backslash \mathcal{B}} < \xi, m > m), u^2 >= \sum_{m \in \mathcal{M}_n \backslash \mathcal{B}} < \xi, m >< dm, u^2 > .$$

This implies that there exists a monomial $m \in \mathcal{M}_n \backslash \mathcal{B}$ such that $< \xi, m >\neq 0$ and $< dm, u^2 >\neq 0$. The only possible monomial is $[u|u]$. It follows that $< \xi, [u|u] >\neq 0$. We have

$$\begin{aligned} 0 &= < de, [xyxy|u] > \\ &= < d\xi, [xyxy|u] > \\ &= < d(\sum_{m \in \mathcal{M}_n} < \xi, m > m), [xyxy|u] > \\ &= \sum_{m \in \mathcal{M}_n} < \xi, m >< dm, [xyxy|u] > . \end{aligned}$$

It follows that

$$\sum_{m \neq [u|u]} < \xi, m >< dm, [xyxy|u] >= - < \xi, [u|u] >< d[u|u], [xyxy|u] >\neq 0.$$

There exists thus a monomial $m \neq [u|u]$ such that $< \xi, m > \neq 0$ and $< dm, [xyxy|u] > \neq 0$. The possible monomials are $[x|yxy|u]$, $[xy|xy|u]$, and $[xyx|y|u]$. We thus are in one of the following situations:

1) $< \xi, [x|yxy|u] > \neq 0$,

2) $< \xi, [xy|xy|u] > \neq 0$,

3) $< \xi, [xyx|y|u] > \neq 0$.

Suppose that we are in the first situation. Then we have

$$\begin{aligned} 0 &= < de, [x|yxy|xyxy] > \\ &= < d\xi, [x|yxy|xyxy] > \\ &= < d(\sum_{m \in \mathcal{M}_n} < \xi, m > m), [x|yxy|xyxy] > \\ &= \sum_{m \in \mathcal{M}_n} < \xi, m >< dm, [x|yxy|xyxy] > . \end{aligned}$$

It follows that

$$\sum_{m \neq [x|yxy|u]} < \xi, m >< dm, [x|yxy|xyxy] >= - < \xi, [x|yxy|u] >< d[x|yxy|u], [x|yxy|xyxy] > \neq 0.$$

There exists thus a monomial $m \neq [x|yxy|u]$ such that

$$< \xi, m > \neq 0 \qquad \text{and} \qquad < dm, [x|yxy|xyxy] > \neq 0.$$

The possible monomials are $m_1 = [x|y|xy|xyxy]$, $m_2 = [x|yx|y|xyxy]$, $m_3 = [x|yxy|x|yxy]$, $m_4 = [x|yxy|xy|xy]$, and $m_5 = [x|yxy|xyx|y]$. It follows that there exists an index $i \in \{1, \ldots, 5\}$ such that

$$< \sigma e, m_i >=< e - \xi, m_i >=< e, m_i > - < \xi, m_i >= - < \xi, m_i > \neq 0.$$

As each $m_i \in T^4(s\overline{TV})$, this implies that $\sigma e \notin B_3TV$ and thus that $n \geq 4$. The proof of the fact that $n \geq 4$ in the other two situations is analogous. □

By 2.6, we have the following corollary:

Corollary 3.3. *The Milnor-Moore spectral sequence of $Z \cup e^{16}$ satisfies $E^{\infty}_{3,*} = 0$ and $E^{\infty}_{4,*} \neq 0$.* □

Remark 3.4. Let Z_0 be a rationalization of the space Z. Then we have $\mathrm{e}_0(Z_0) = 2$ and $\mathrm{cat}(Z_0) = 3$. Indeed, we have $\mathrm{e}_0(Z_0) = \mathrm{e}_0(Z) = 2$ and, as Z_0 is the homotopy cofibre of a map into a two-cone, $\mathrm{cat}(Z_0) \leq 3$. The equality $\mathrm{cat}(Z_0) = 3$ holds since for any rationalization $\gamma_0 : S^{15}_0 \to Z_0$ of γ

$$4 = \mathrm{e}_0(Z \cup e^{16}) = \mathrm{e}_0(Z_0 \cup_{\gamma_0} CS^{15}_0) \leq \mathrm{cat}(Z_0 \cup_{\gamma_0} CS^{15}_0) \leq \mathrm{cat}(Z_0) + 1 \leq 3 + 1 = 4.$$

As cat increases by at most one when a cell is attached to a space, it was, of course, *a priori* clear that the space Z_0 would satisfy $\mathrm{e}_0(Z_0) < \mathrm{cat}(Z_0)$. It should be noticed here that Toomer originally conjectured that e_0 equals cat for rational spaces. The first counterexample to this conjecture has been given by J.-M. Lemaire and F. Sigrist who showed that the rationalization Y_0 of the CW-complex $Y = S^2 \vee \mathbb{C}P^2 \cup e^7$, where the cell e^7 is attached by the Whitehead product of S^2 and the Hopf map $S^5 \to \mathbb{C}P^2$, satisfies $\mathrm{e}_0(Y_0) = 2$ and $\mathrm{cat}(Y_0) = 3$ [11].

4 The integral Toomer invariant and the σ^i-category

In [16] the following sequence of approximations of the L.-S. category has been introduced:

Definition 4.1. Let X be any space and $i \geq 1$ be an integer. The σ^i-category of X, $\sigma^i\mathrm{cat}(X)$, is the least integer n such that the i-fold suspension of the nth Ganea fibration $\Sigma^i g_n(X) : \Sigma^i G_n(X) \to \Sigma^i X$ has a homotopy section. If no such n exists, one sets $\sigma^i\mathrm{cat}(X) = \infty$. The σ-category of X is the (possibly infinite) number $\sigma\mathrm{cat}(X) = \inf_{i\in\mathbb{N}} \sigma^i\mathrm{cat}(X)$.

The $\sigma^i\mathrm{cat}(X)$ form a decreasing sequence whose first term $\sigma^1\mathrm{cat}$ coincides for path-connected spaces with the weak category G-wcat(X) in the sense of Gilbert [7] and whose limit $\sigma\mathrm{cat}(X)$ coincides for path-connected finite CW-complexes with Rudyak's strict category weight [12]. As the existence of a homotopy section for $\Sigma^i g_n(X) : \Sigma^i G_n(X) \to \Sigma^i X$ implies the existence of a section for $H_*(g_n(X))$, we have $\mathrm{e}(X) \leq \sigma^i\mathrm{cat}(X)$. This inequality fits into the following sequence of inequalities where $i \geq j$ and X is path-connected:

$$\mathrm{e}_0(X) \leq \mathrm{e}(X) \leq \sigma\mathrm{cat}(X) \leq \sigma^j\mathrm{cat}(X) \leq \sigma^i\mathrm{cat}(X) \leq \sigma^1\mathrm{cat}(X) \leq \mathrm{cat}(X).$$

When X is a simply connected rational space we have $\mathrm{e}_0(X) = \mathrm{e}(X) = \sigma\mathrm{cat}(X) = \sigma^i\mathrm{cat}(X) = \sigma^1 cat(X) =$ G-wcat(X). The $\sigma^i\mathrm{cat}$ can therefore be interpreted as topological versions of the Toomer invariant. We can easily deduce from the result of the previous section that the invariants e, σcat, σ^icat, and G-wcat may increase by more than 1 when a cell is attached to a rationalization Z_0 of the space Z constructed in section 3. However, Z_0 is not a finite CW-complex and it has seemed interesting to us to show that such a phenomenon can also occur when all the spaces are finite CW-complexes.

Theorem 4.2. *Let Z and $Z \cup e^{16}$ be the CW-complexes defined in section 3. Then for any $i \geq 1$*

$$e(Z) = \sigma^i cat(Z) = 2 \quad and \quad e(Z \cup e^{16}) = \sigma^i cat(Z \cup e^{16}) = 4.$$

In particular, G-wcat$(Z) = \sigma cat(Z) = 2$ and G-wcat$(Z \cup e^{16}) = \sigma cat(Z \cup e^{16}) = 4$.

Proof: The equalities $\mathrm{e}(Z \cup e^{16}) = 4$ and $\sigma^i\mathrm{cat}(Z \cup e^{16}) = 4$ follow from $\mathrm{e}_0(Z \cup e^{16}) = \mathrm{cat}(Z \cup e^{16}) = 4$ and from the inequalities $\mathrm{e}_0 \leq \mathrm{e} \leq \sigma^i\mathrm{cat} \leq \mathrm{cat}$. To obtain the remaining equalities it suffices to prove that $\sigma^i\mathrm{cat}(Z) \leq 2$. This follows from the construction of the space Z and from the following proposition. □

Proposition 4.3. *Let X be any space and $i \geq 1$ be an integer. Suppose that $\sigma^i cat(X) \leq k$ with $k \geq 2$ and consider a map $\omega : S^{p+n-1} \to X$ representing the Whitehead product of two homotopy classes $\alpha \in \pi_p(X)$ and $\beta \in \pi_n(X)$ $(p, n \geq 1)$. Then $\sigma^i cat(X \cup_\omega D^{p+n}) \leq k$.*

In [13] H. Scheerer, D. Stanley, and D. Tanré show that the σ-category of the Lemaire-Sigrist space $S^2 \vee CP(2) \cup e^7$ (cf. 3.4) is 2. The proof of Proposition 4.3 is a generalization of their computation. Before we give this proof we fix some notations. We denote by $\bar{\Omega}X$ the ordinary loop space of the space X. For a map $f : S^p \to X$ $(p \geq 1)$ we denote by $f^\sharp : S^{p-1} \to \bar{\Omega}X$ the adjoint map. Recall that there exists a natural homotopy equivalence $\lambda_1 : \Sigma\bar{\Omega}X \to G_1(X)$ such that $g_1(X) \circ \lambda_1 = ev$ where $ev : \Sigma\bar{\Omega}X \to X$ is the evaluation map. For $k \geq 2$ denote by λ_k the natural map $\Sigma\bar{\Omega}X \to G_k(X)$ which is the composition of λ_1 and the map $G_1(X) \to G_k(X)$ coming from the Ganea construction. For any cofibration sequence $S^p \xrightarrow{f} X \xrightarrow{j} Y = X \cup_f D^{p+1}$ the map $\Sigma\bar{\Omega}j \circ \Sigma f^\sharp = \Sigma(j \circ f)^\sharp$ is homotopically trivial and so is each composition $S^p \xrightarrow{\lambda_n \circ \Sigma f^\sharp} G_n(X) \xrightarrow{G_n(j)} G_n(Y)$. Using the adjunction, we can construct an extension $\chi : D^{p+1} \to \Sigma\bar{\Omega}X$ of the map $\Sigma\bar{\Omega}j \circ \Sigma f^\sharp$ such that $ev \circ \chi : D^{p+1} \to Y = X \cup_f D^{p+1}$ is the canonical map. Then for each $n \geq 1$ the composite $\lambda_n \circ \chi$ is an extension of the composition $S^p \xrightarrow{\lambda_n \circ \Sigma f^\sharp} G_n(X) \xrightarrow{G_n(j)} G_n(Y)$ and $g_n(Y) \circ \lambda_n \circ \chi : D^{p+1} \to Y$ is the canonical map. By the universal property of pushouts, we obtain thus a map $\tilde{g}_n : G_n(X) \cup_{\lambda_n \circ \Sigma f^\sharp} D^{p+1} \to Y$ which sends D^{p+1} identically to D^{p+1} and a commutative diagram in which $\varphi_n \circ j_n = G_n(j)$:

$$\begin{array}{ccccc}
G_n(X) & \xrightarrow{j_n} & G_n(X) \cup_{\lambda_n \circ \Sigma f^\sharp} D^{p+1} & \xrightarrow{\varphi_n} & G_n(Y) \\
\downarrow{\scriptstyle g_n(X)} & & \downarrow{\scriptstyle \tilde{g}_n} & & \downarrow{\scriptstyle g_n(Y)} \\
X & \xrightarrow[j]{} & Y & = & Y.
\end{array}$$

In order to obtain a homotopy section of $g_n(Y)$ (resp. $\Sigma^i g_n(Y)$) it suffices thus to construct a homotopy section of $\tilde{g}_n$ (resp. $\Sigma^i\tilde{g}_n$).

Proof of Proposition 4.3: In what follows we fix a representative $[\iota_p,\iota_n] : S^{p+n-1} \to S^p \vee S^n$ of the Whitehead product of the generators of $\pi_p(S^p)$ and $\pi_n(S^n)$ and for two maps $\phi : S^p \to W$ and $\gamma : S^n \to W$ we denote by $[\phi,\gamma]$ the composition $S^{p+n-1} \overset{[\iota_p,\iota_n]}{\to} S^p \vee S^n \overset{\phi\vee\gamma}{\to} W \vee W \overset{\nabla}{\to} W$ where ∇ is the folding map. Let $f : S^p \to X$ and $g : S^n \to X$ be representatives of the classes α and β. The hypothesis $\sigma^i\mathrm{cat}(X) \le k$ means that the map $\Sigma^i g_k(X)$ admits a homotopy section $\sigma : \Sigma^i X \to \Sigma^i G_k(X)$. We will use σ to construct a homotopy section of the map

$$\Sigma^i\tilde{g}_k : \Sigma^i(G_k(X) \cup_{\lambda_k\circ\Sigma[f,g]^\sharp} D^{p+n}) \to \Sigma^i(X \cup_{[f,g]} D^{p+n}).$$

In order to do this we first show that $\lambda_k \circ \Sigma[f,g]^\sharp \simeq [\lambda_k \circ \Sigma f^\sharp, \lambda_k \circ \Sigma g^\sharp]$: By adjointness and the naturality of the Whitehead product, we have $ev \circ \Sigma[f,g]^\sharp = [f,g]$ and $ev \circ [\Sigma f^\sharp, \Sigma g^\sharp] = [ev \circ \Sigma f^\sharp, ev \circ \Sigma g^\sharp] = [f,g]$. Therefore $g_1(X) \circ \lambda_1 \circ (\Sigma[f,g]^\sharp - [\Sigma f^\sharp, \Sigma g^\sharp]) \simeq *$ and the map $\lambda_1 \circ (\Sigma[f,g]^\sharp - [\Sigma f^\sharp, \Sigma g^\sharp])$ lifts up to homotopy to $F_1(X)$. As the map $G_1(X) \to G_k(X)$ factors for $k \ge 2$ through $G_1(X) \cup_{F_1(X)} CF_1(X)$, it follows that $\lambda_k \circ (\Sigma[f,g]^\sharp - [\Sigma f^\sharp, \Sigma g^\sharp])$ is homotopically trivial. We have thus $\lambda_k \circ \Sigma[f,g]^\sharp \simeq \lambda_k \circ [\Sigma f^\sharp, \Sigma g^\sharp] = [\lambda_k \circ \Sigma f^\sharp, \lambda_k \circ \Sigma g^\sharp]$.

As $i \ge 1$, the map $\Sigma^i[\lambda_k \circ \Sigma f^\sharp, \lambda_k \circ \Sigma g^\sharp]$ is homotopically trivial. It follows that the map $\Sigma^i(\lambda_k \circ \Sigma[f,g]^\sharp)$ is homotopically trivial and thus that the map $\Sigma^i\tilde{g}_k : \Sigma^i(G_k(X) \cup_{\lambda_k\circ\Sigma[f,g]^\sharp} D^{p+n}) \to \Sigma^i(X \cup_{[f,g]} D^{p+n})$ is weakly equivalent to the map $\Sigma^i g_k(X) \vee id : \Sigma^i G_k(X) \vee S^{n+p+i} \to \Sigma^i X \vee S^{n+p+i}$. This map admits $\sigma \vee id$ as a homotopy section and this implies that $\sigma^i\mathrm{cat}(X \cup_{[f,g]} D^{p+n}) \le k$. □

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