An axiomatic approach to Lusternik-Schnirelmann category type homotopy invariants

Thomas Kahl

1 Introduction

The Lusternik-Schnirelmann category of a topological space X, denoted $\operatorname{cat} X$, is defined to be the least integer n such that X can be covered by $n+1$ open sets each of which is contractible in X. (If no such n exists one sets $\operatorname{cat} X = \infty$.) Category was originally introduced by Lusternik and Schnirelmann ([11]) (who considered closed instead of open coverings) to estimate the minimal number of critical points of differentiable functions on manifolds. They showed that a smooth function on a smooth compact manifold M admits at least $\operatorname{cat} M + 1$ critical points. The Lusternik-Schnirelmann category has received an enormous amount of investigation over the years. For a survey, the reader is refered to the article of I. M. James ([9]).

Examining the Lusternik-Schnirelmann category, topologists have been motivated to define related numerical homotopy invariants (in fact, it is easy to see that cat is a homotopy invariant). Examples of such invariants are the strong category (cf. [6]), the weak category (cf. [7]), and Toomer's invariant (cf. [13]). Another invariant, the cone-length of a topological space, was introduced by Cornea. Recently, he showed that it coincides with the strong category ([4]). In general, the different invariants do

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not coincide, but they are related in so far as there are several inequalities and in so far as they have a lot of common properties. On this background one could ask, in how far the theory of these invariants can be developed based on a general concept of "Lusternik-Schnirelmann category type homotopy invariants". Such an approach could be useful due to the fact that statements would always hold for whole classes of invariants. If for instance a certain formula turned out to be the consequence of certain principles which also hold for other invariants than those, for which the formula is known, the formula would be proven for these other invariants. The general approach could also turn out to be useful if it was possible to find effective characterizations of the different invariants within the general framework and so to obtain criteria which allow to decide whether conjectures on the coincidence of invariants hold or not.

In this text a bit of the theory of a few of the mentioned invariants will be developed in an axiomatic framework. It is not the aim of this text to present a definitive general theory of Lusternik-Schnirelmann category type homotopy invariants. This text is a first attempt to approach the theory of such invariants from an axiomatic point of view.

2 Category, strong category, and cone-length

We will work in a pointed (proper) J-category $\mathcal{C}$, i.e. in a cofibration and fibration category in the sense of Baues which possesses a zero object 0 and satisfies the cube axiom of Doeraene. A J-category is hence a category endowed with weak equivalences ($\xrightarrow{\sim}$), cofibrations ($\rightarrowtail$), and fibrations ($\twoheadrightarrow$) such that several axioms hold. The category of pointed spaces and basepoint preserving continuous maps is a J-category if weak equivalences are free homotopy equivalences, cofibrations are closed (free) cofibrations, and fibrations are (free) fibrations (see [5]). The reader can hence assume that what is presented takes place in this category. It applies, however, also to many other categories which are important in algebraic topology. The axioms and examples of J-categories can be found in [5], [8], and [10]. Some fundamental definitions (suspensions, joins etc.) are recalled in the appendix.

There are two alternative definitions of the Lusternik-Schnirelmann category of a space, one due to G. Whitehead, the other due to Ganea. In his thesis ([5]), Doeraene established versions of these definitions in J-categories and showed their coincidence in the abstract setting. Later Hess and Lemaire ([8]) gave an abstract definition of the category of a map (in the sense of Berstein and Ganea (see [2])) and showed that it coincides for objects with Doeraene's definitions. The definition of Hess and Lemaire reads as follows:

Definition 2.1. *For a map $f: X \longrightarrow Y$ and $n \in$ set $\operatorname{cat} f \leq n$ if*

$(n = 0)$ f is nulhomotopic (i.e. the image $\{f\}$ of f under the localization functor $\mathcal{C} \longrightarrow \operatorname{Ho}\mathcal{C}$ ($\operatorname{Ho}\mathcal{C} =$ the homotopy category) is the zero map $0: X \longrightarrow 0 \longrightarrow Y$),

$(n > 0)$ there is a homotopy push-out (in the sense of Baues ([1]))

$$\begin{array}{ccc} U & \longrightarrow & W \\ \big\downarrow & \text{h.p.o.} & \big\downarrow w \\ V & \underset{v}{\longrightarrow} & X \end{array}$$

such that $\operatorname{cat} fw \leq n-1$ and $\operatorname{cat} fv = 0$.

The least n such that $\operatorname{cat} f \leq n$ is called the category *of f and is denoted $\operatorname{cat} f$. If no such n exists one sets $\operatorname{cat} f := \infty$. The category of an object X, $\operatorname{cat} X$, is defined to be $\operatorname{cat} \operatorname{id}_X$.*

Remarks 2.2. (i) A priori there are two meanings of the phrase "$\operatorname{cat} f \leq n$". The first is that there is a homotopy push-out

$$\begin{array}{ccc} U & \longrightarrow & W \\ \big\downarrow & \text{h.p.o.} & \big\downarrow{\scriptstyle w} \\ V & \xrightarrow[v]{} & X \end{array}$$

such that $\operatorname{cat} fw \leq n-1$ and $\operatorname{cat} fv = 0$. The second is that there is a $m \leq n$ and a homotopy push-out

$$\begin{array}{ccc} U & \longrightarrow & W \\ \big\downarrow & \text{h.p.o.} & \big\downarrow{\scriptstyle w} \\ V & \xrightarrow[v]{} & X \end{array}$$

such that $\operatorname{cat} fw \leq m-1$ and $\operatorname{cat} fv = 0$. To see that the two meanings coincide one has to show that – in the first sense – $\operatorname{cat} f \leq n$ implies $\operatorname{cat} f \leq n+1$ (which is trivial in the second sense since $n \leq n+1$). We leave the simple proof to the reader.

(ii) For a basepoint-preserving continuous map $f\colon X \longrightarrow Y$ where X is normal and well-pointed and Y is path-connected, $\operatorname{cat} f$ turns out to be the least n such that X can be covered by $n+1$ open sets on each of which f is nulhomotopic (rel. the basepoint or not) (cf. [10]). This is the original definition of the category of a map due to Berstein and Ganea. For normal and path-connected well-pointed spaces we obtain hence cat as defined in the introduction.

The definition of Hess and Lemaire is equivalent to the definitions of Doeraene. This means that cat can be characterized with the following Ganea-construction:

Definition 2.3. *A map $g^nB\colon G^nB \longrightarrow B$ is called a n^{th}* Ganea-map *of B if*

$(n=0)$ *G^nB is weakly equivalent to 0, $G^0B \quad 0$,* [1]

[1]Two objects X and X' are said to be weakly equivalent ($X \quad X'$) if they are connected by a finite sequence of weak equivalences

$$X \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \ldots \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow X'.$$

$(n > 0)$ *there is a* $n-1^{st}$ *Ganea-fibration* $F \to G^{n-1}B \overset{g^{n-1}B}{\twoheadrightarrow} B$ *such that* g^nB *is weakly equivalent over* B *to the map* $g^{n-1}B + 0 \colon G^{n-1}B \vee_F CF \longrightarrow B$ *obtained when a cone* CF *is attached on the fibre* F *and mapped to* B *via the zero object.*

If $G^nB \longrightarrow B$ *is a* n^{th} *Ganea-map of* B*, then* G^nB *is called a* n^{th} *Ganea-object of* B*.*

Note that all n^{th} Ganea-maps of B are weakly equivalent over B and that a map which is weakly equivalent over B to a n^{th} Ganea-map of B is itself a n^{th} Ganea-map of B.

Theorem 2.4. $\operatorname{cat} X \leq n$ *iff a(ll)* n^{th} *Ganea-map(s) of* X *has (have) a section in the homotopy category, i.e. iff for a(ll)* n^{th} *Ganea-map(s)* g^nX *of* $X\{g^nX\}$ *has a section.*

For a proof the reader is refered to [8] or [10]. If one takes the theorem as a definition of category one obtains one of the (equivalent) definitions of Doeraene.

In [6], Ganea defined the strong category of a space X, $\operatorname{Cat} X$, to be the least integer n such that X has the homotopy type of a CW-complex which can be covered by $n+1$ self-contractible subcomplexes. He showed that there is an equivalent definition using cofibration sequences. This definition has an analogue in J-categories:

Definition 2.5. *For* $X \in \mathcal{C}$ *set* $\operatorname{Cat} X \leq n$ *if*

$(n = 0)$ $X \quad 0$,

$(n > 0)$ *there is a cofibration* $F \rightarrowtail E \longrightarrow E/F$ *such that* $\operatorname{Cat} E \leq n-1$ *and* $X \quad E/F$.

There are canonical weak equivalences in the categories $\mathcal{C}(2)$, $\mathcal{C}(4)$, and $\mathcal{C}_B$ of maps, commutative squares, resp. maps with target B. So it makes sense to talk of weakly equivalent maps and commutative squares. Maps with the same target B which are weakly equivalent in $\mathcal{C}_B$ are said to be weakly equivalent over B.

The least n such that $\operatorname{Cat} X \leq n$ *is called the* strong category *of* X*. If no such n exists one sets* $\operatorname{Cat} X := \infty$.

As mentioned in the introduction, it is a recent result of Cornea that the strong category of a space is the same as what he defined before to be its cone-length ([4]). In J-categories the definition of cone-length reads:

Definition 2.6. *For $X \in \mathcal{C}$ set* $\operatorname{Cl} X \leq n$ *if*

$(n = 0)$ $X \quad 0$,

$(n > 0)$ *there is a cofibration* $F \rightarrowtail E \longrightarrow E/F$ *such that* $\operatorname{Cl} E \leq n-1$, $X \quad E/F$, *and F is a $(n-1)$-fold suspension of some object Z,* $F = \Sigma^{n-1} Z$ *(where* $A = \Sigma^0 B$ *iff* $A \quad B$*).*

The least n such that $\operatorname{Cl} X \leq n$ *is called the* cone-length *of X. If no such n exists one sets* $\operatorname{Cl} X := \infty$.

Remark 2.7. A space X has category ≤ 1 iff it is a co-H-space. This follows immediately from Whitehead's characterization of category. Since $\operatorname{Cat} X \leq 1$ iff $\operatorname{Cl} X \leq 1$ iff X has the homotopy type of a suspension, the three invariants do not in general coincide. We will, however, show in (3.8) that Cl always equals Cat.

3 LS-invariants

The similarity of cat and Cat is obvious. Both are (more or less) defined to be the minimal cardinality of a covering consisting of sets satisfying a contractibility condition. There is, however, also an obvious difference. While the contractibility condition for cat is a relation between two objects – a set is contractible in a space –, the one for Cat is a property of one object – a set is contractible in itself. This difference is reflected in the fact that the abstract definition of the category of an object refers to the definition of the category of a map while we did not define any notion of the strong category of a map. In order to grasp axiomatically what cat and Cat have in common it is, however, opportune to define the ***strong category of a map*** $f\colon X \longrightarrow Y$, $\operatorname{Cat} f$, to be $\operatorname{Cat} X$. We will now define a notion of "Lusternik-Schnirelmann category type invariant", shortly "LS-invariant", that covers cat and Cat.

Definition 3.1. *A map* $c\colon \mathcal{M}\quad(\mathcal{C}) \longrightarrow \quad \cup\{\infty\}$ *is called a* LS-invariant *if the following conditions hold:*

A1 *If f and g are weakly equivalent in the category of the morphisms of $\mathcal{C}$, $f \quad g$, then $c(f) = 0$ iff $c(g) = 0$.*

A2 $c(0 \longrightarrow B) = 0$.

A3 *For $n > 0$ and a map $f\colon X \longrightarrow Y$, $c(f) \leq n$ iff there is a homotopy push-out*

$$\begin{array}{ccc} U & \longrightarrow & W \\ \big\downarrow & \text{h.p.o.} & \big\downarrow{\scriptstyle w} \\ V & \underset{v}{\longrightarrow} & X \end{array}$$

such that $c(fw) \leq n-1$ and $c(fv) = 0$.

A4 *For a fibration $F \longrightarrow E \twoheadrightarrow B$ and a map $f\colon A \longrightarrow B$ with $c(f) = 0$, $c(\bar{f}\colon (A \times_B E) \vee_F CF \underset{\mathrm{pr}_B \vee_{\mathrm{id}} \mathrm{id}}{\longrightarrow} E \vee_F CF) = 0$.*

A5 $c(f) = 0 \Rightarrow c(g \circ f) = 0$.

A6 $c(f) = 0 \Rightarrow \{f\} = 0$.

The axiom A3 says that for a LS-invariant c and a map $f\colon X \longrightarrow Y$, $c(f)$ is defined to be the minimal cardinality of a "covering up to homotopy" of X by objects such that the "restrictions" of f to these objects satisfy the condition $c = 0$. A LS-invariant is hence determined by the definition of the statement $c = 0$. Some conditions on such definitions are given by the other axioms.

The proof of the following proposition is a simple induction and is left to the reader.

Proposition 3.2. *Let* $c\colon \mathcal{M}\quad(\mathcal{C}) \longrightarrow \quad \cup\{\infty\}$ *be a LS-invariant.*
(i) $f \quad g \Rightarrow c(f) = c(g)$.
(ii) $c(g \circ f) \leq c(f)$.

Hence, by (3.2)(i), a LS-invariant is indeed an invariant. Note that by (3.2)(ii), $c(A \longrightarrow B) \leq c(\mathrm{id}_A)$.

Proposition 3.3. cat *and* Cat *are LS-invariants.*

Proof: (a) cat. A1, A2, A3, A5, and A6 are obvious. For A4 let $F \longrightarrow E \twoheadrightarrow B$ be a fibration and $f\colon A \longrightarrow B$ be a map such that $\mathrm{cat}(f) = 0$. Construct a diagram

$$\begin{array}{ccccccccc}
F & = & F & = & F & \longrightarrow & F' & = & F' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
L \times_B E & \twoheadrightarrow & A \times_B E & \longrightarrow & E & \longrightarrow & N & \twoheadleftarrow & PM \times_B N \\
\downarrow\!\!\downarrow & \text{p.b.} & \downarrow\!\!\downarrow & \text{p.b.} & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & \text{p.b.} & \downarrow\!\!\downarrow \\
L & \twoheadrightarrow & A & \longrightarrow & B & \longrightarrow & M & \twoheadleftarrow & PM \longleftarrow 0
\end{array}$$

such that L is cofibrant[2], M is 0-fibrant, and the columns are fibrations. Since $\{f\} = 0$, $L \longrightarrow M$ factors over PM and hence $L \times_B E \to N$ factors over $PM \times_M N$. Set $CF' = CF \vee_F F'$ to construct the following diagram:

[2] An object X is cofibrant if any "trivial fibration" $Y \twoheadrightarrow X$ has a section. X is called 0-cofibrant if $0 \longrightarrow X$ is a cofibration. If any trivial cofibration $X \rightarrowtail Y$ admits a retraction, then X is called fibrant. If $X \longrightarrow 0$ is a fibration, then X is said to be 0-fibrant. In the mentioned J-category of pointed spaces all objects are (0−) fibrant. The (0−) cofibrant objects are the well-pointed spaces.

$$\begin{array}{ccc} (L \times_B E) \vee_F CF \longrightarrow (A \times_B E) \vee_F CF & \xrightarrow{\bar{f}} & E \vee_F CF \\ \downarrow & & \downarrow \\ (PM \times_M N) \vee_{F'} CF' & \longrightarrow & N \vee_{F'} CF'. \end{array}$$

Since $CF' \quad (PM \times_M N) \vee_{F'} CF', \{\bar{f}\} = 0$. Thus, $\operatorname{cat} \bar{f} = 0$.

(b) Cat. A1, A2, A5, and A6 are obvious. For the proof of A3 let first $\operatorname{Cat}(f\colon A \longrightarrow B) \leq n$ $(n > 0)$. Then there exists a cofibration $F \rightarrowtail E \longrightarrow E/F$ such that $E/F \quad A$ and $\operatorname{Cat} E \leq n-1$. Obviously, there is a homotopy push-out

$$\begin{array}{ccc} U & \longrightarrow & W \\ \downarrow & \text{h.p.o.} & \downarrow w \\ V & \xrightarrow[v]{} & A \end{array}$$

which is weakly equivalent to the homotopy push-out

$$\begin{array}{ccc} F & \rightarrowtail & E \\ \downarrow & \text{h.p.o.} & \downarrow \\ 0 & \rightarrowtail & E/F. \end{array}$$

Thus, $\operatorname{Cat}(fv) = \operatorname{Cat}(V \quad 0) = 0$ and $\operatorname{Cat}(fw) = \operatorname{Cat}(W) = \operatorname{Cat}(E) \leq n-1$. Suppose conversely that there is a homotopy push-out $U - W - A - V$ such that $\operatorname{Cat}(fw) = \operatorname{Cat}(W) \leq n-1$ and $\operatorname{Cat}(fv) = \operatorname{Cat}(V) = 0$. Choose a factorization $U \rightarrowtail E \to W$. Then there is a weak equivalence $E \vee_U V \longrightarrow A$. Since $\operatorname{Cat}(V) = 0$, there is a weak equivalence $V \longrightarrow 0$ and hence a weak equivalence $E \vee_U V \longrightarrow E/U$. Thus, there is a cofibration $U \rightarrowtail E \longrightarrow E/U \quad A$ such that $\operatorname{Cat} E \leq n-1$. Thus, $\operatorname{Cat} f = \operatorname{Cat} A \leq n$.

For A4 let $F \longrightarrow E \longrightarrow\!\!\!\!\!\rightarrow B$ be a fibration and $f\colon A \longrightarrow B$ be a map such that $\mathrm{Cat}(f) = 0$. Then there is a weak equivalence $0 \longrightarrow A$. Thus, there is a weak equivalence $F \longrightarrow A \times_B E$ and hence a weak equivalence $CF \longrightarrow (A \times_B E) \vee_F CF$. Thus, $\mathrm{Cat}(\bar{f}) = 0$.

We considered functions which are defined for maps in our definition of LS-invariants because cat is a relative concept. As Cat is not, it is particularly interesting to study the object-functions of LS-invariants, i.e. their restrictions to identities. We will do this by defining a notion of LS-invariant for functions defined on $\mathcal{O}\ \mathcal{C}$ and showing that the object-function of a LS-invariant is such a LS-invariant. After having established this we will only consider LS-invariants in the new sense so that no confusion will be caused by the unprecise terminology.

Definition 3.4. *A map* $c\colon \mathcal{O}\ (\mathcal{C}) \longrightarrow \quad \cup \{\infty\}$ *is called a* LS-invariant *if the following conditions hold:*

LS1 $X \quad Y \Rightarrow c(X) = c(Y)$.

LS2 $c(0) = 0$.

LS3 *For a map* $f\colon X \longrightarrow Y$, $c(C_f = Y \vee_X CX) \leq c(Y) + 1$.

Proposition 3.5. *Let* $c\colon \mathcal{M}\ \ (\mathcal{C}) \longrightarrow \quad \cup\{\infty\}$ *be a LS-invariant. Then* $c\colon \mathcal{O}\ (\mathcal{C}) \longrightarrow \quad \cup \{\infty\}$, $c(X) := c(\mathrm{id}_X)$, *is a LS-invariant. Moreover, the following statements hold:*
(i) *For a fibration* $F \longrightarrow E \longrightarrow\!\!\!\!\!\rightarrow B$, $c(E \vee_F CF) \leq c(B)$.
(ii) $\mathrm{cat} \leq c \leq \mathrm{Cat}$.

Proof: LS1 and LS2 are obvious.

LS3: Consider

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \text{(h.)p.o.} & \downarrow w \\ CX & \xrightarrow[v]{} & C_f\,. \end{array}$$

Then, $c(\mathrm{id}_{C_f}\, v) = c(v) = (3.2)(\mathrm{i})\ c(0 \longrightarrow C_f) = (\mathrm{A}2)\ 0$. Thus, $c(C_f) = c(\mathrm{id}_{C_f}) \leq c(\mathrm{id}_{C_f}\, w) + 1 = c(w) + 1 \leq (3.2)(\mathrm{ii})\ c(Y) + 1$.

(i): Let $F \longrightarrow E \twoheadrightarrow B$ be a fibration. Consider diagrams

$$\begin{array}{ccc} (A \times_B E) \vee_F CF & \xrightarrow{\bar{f}} & E \vee_F CF \\ \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & B\,. \end{array}$$

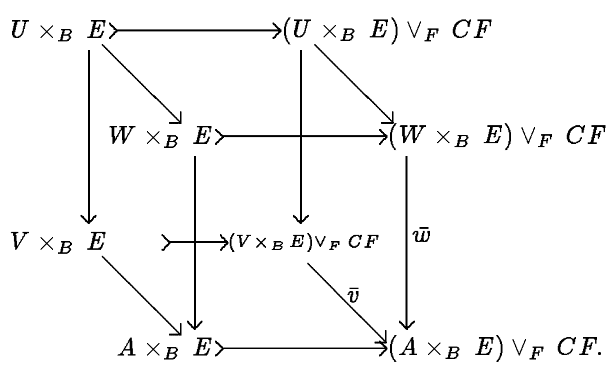
We show inductively $c(f) \leq n \Rightarrow c(\bar{f}) \leq n$. Setting $f = \mathrm{id}_B$, we get (i). If $c(f) = 0$, then $c(\bar{f}) = 0$ by A4. Let $n > 0$ and $c(f) \leq n$. By A3, there is a homotopy push-out

$$\begin{array}{ccc} U & \longrightarrow & W \\ \downarrow & \text{h.p.o.} & \downarrow{\scriptstyle w} \\ V & \xrightarrow[v]{} & A \end{array}$$

such that $c(fw) \leq n - 1$ and $c(fv) = 0$. Since $\mathcal{C}$ is a J-category, we get a homotopy push-out

$$\begin{array}{ccc} U \times_B E & \longrightarrow & W \times_B E \\ \downarrow & \text{h.p.o.} & \downarrow \\ V \times_B E & \longrightarrow & A \times_B E\,. \end{array}$$

Consider



By composition, the right hand square is a homotopy push-out. Inductively, $c(\bar{f}\bar{v}) = 0$ and $c(\bar{f}\bar{w}) \leq n-1$. Thus, by A3, $c(\bar{f}) \leq n$.

(ii): For maps $f\colon X \longrightarrow Y$ we show inductively, that $\operatorname{cat}(f) \leq n$ if $c(f) \leq n$ (resp. $c(f) \leq n$ if $\operatorname{Cat} f \leq n$).

$n = 0$: $c(f) = 0 \Rightarrow$ (A6) $\{f\} = 0 \Rightarrow \operatorname{cat} f = 0$ (resp. $\operatorname{Cat} f = 0 \Longleftrightarrow f \quad 0 \longrightarrow Y \Rightarrow$ (A1) $c(f) = c(0 \longrightarrow Y) =$ (A2) 0).

$n > 0$: Let $c(f) \leq n$ (resp. $\operatorname{Cat} f \leq n$). By A3 (resp. (3.3) and A3), there is a homotopy push-out

$$\begin{array}{ccc} U & \longrightarrow & W \\ \Big\downarrow & \text{h.p.o.} & \Big\downarrow w \\ V & \underset{v}{\longrightarrow} & X \end{array}$$

such that $c(fw) \leq n-1$ and $c(fv) = 0$ (resp. $\operatorname{Cat} fw \leq n-1$ and $\operatorname{Cat} fv = 0$). Inductively we have $\operatorname{cat}(fv) = 0$ and $\operatorname{cat}(fw) \leq n-1$, and hence $\operatorname{cat}(f) \leq n$ (resp. $c(fv) = 0$ and $c(fw) \leq n-1$, and hence $c(f) \leq n$).

Thus, cat and Cat are LS-invariants in the sense of (3.4). Moreover, cat $\leq$ Cat. Notice that we showed cat $\leq c \leq$ Cat for LS-invariants in the sense of (3.1).

In what follows, unless indicated otherwise, c is a LS-invariant in the sense of (3.4).

We will next show that $c \leq \operatorname{Cat} = \operatorname{Cl} \leq \operatorname{cat} + 1$ and give a characterization of cat as a LS-invariant $\geq$ cat.

If c is the object-function of a (3.1)-LS-invariant, then, by (3.5), $c \leq$ Cat. This inequality holds for arbitrary LS-invariants.

Proposition 3.6. $c \leq$ Cat.

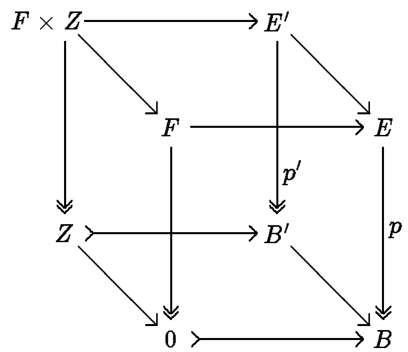
Proof: Let X be any object of $\mathcal{C}$. We show inductively that $c(X) \leq n$ if $\operatorname{Cat} X \leq n$.

If $\operatorname{Cat} X = 0$, then $X \quad 0$. By LS1 and LS2, $c(X) = c(0) = 0$.

If $\operatorname{Cat} X \leq n > 0$, then there is a cofibration $F \rightarrowtail E \to E/F$ such that $X \quad E/F$ and $\operatorname{Cat} E \leq n-1$. Inductively, $c(E) \leq n-1$. By LS2 and LS3, $c(X) = c(E/F) = c(E \vee_F CF) \leq c(E) + 1 \leq n$.

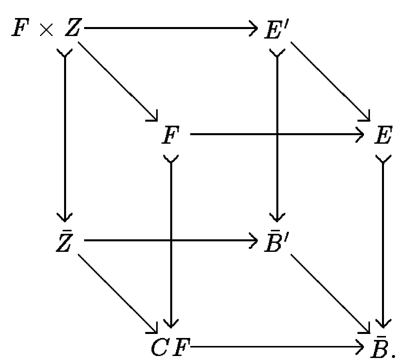
A LS-invariant $\geq$ Cat is hence automatically Cat. As Cl $\geq$ Cat, it suffices to show that Cl is a LS-invariant to obtain that Cl equals Cat. We point out that this is exactly what is done in [4]. As in [4], the proof that Cl is a LS-invariant is based on the following fact:

Proposition 3.7. *Consider*

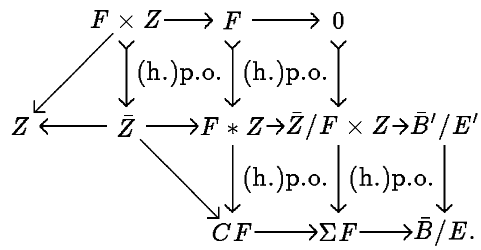


where the bottom is a push-out and the vertical faces are pullbacks. Then there is a cofibration $\Sigma(F \wedge Z) \rightarrowtail M' \longrightarrow M$ *where* $M' \longrightarrow M$ *is weakly equivalent to* $C_{p'} \longrightarrow C_p$.

Proof: Since we are working in a J-category, the top of the cube is a homotopy push-out. Consider the cube as a map of commutative squares and convert this map into a cofibration:



Now consider the following diagram:



The right hand square is a homotopy push-out because the bottom and the top of the above cube are homotopy push-outs. The assertion follows because $\Sigma(F \wedge Z) = F * Z$ (see appendix) and $\bar{B}'/E' \longrightarrow \bar{B}/E$ is weakly equivalent to the map between the homotopy cofibres of p' and p.

Cornea gives a different proof of (3.7). It seems possible to use his arguments in the abstract context, too.

Theorem 3.8. Cl *is a LS-invariant. Hence, by* (3.6), Cl = Cat.

Proof: LS1 and LS2 are clear. For LS3 we show inductively that for any map $f: X \longrightarrow Y$, if $\mathrm{Cl}(Y) \leq n$, then $\mathrm{Cl}(C_f) \leq n+1$.

$n = 0$: If $\mathrm{Cl}(Y) = 0$, then $Y \quad 0$. Thus, we have the cofibration $X \rightarrowtail CX \longrightarrow \Sigma X \quad C_f$. Since $\mathrm{Cl}(CX) = 0$, $\mathrm{Cl}(C_f) \leq 1$.

$n > 0$: If $\mathrm{Cl}(Y) \leq n$, then there is a cofibration $Z \rightarrowtail B' \longrightarrow B$ such that $B \quad Y$, $\mathrm{Cl}(B') \leq n-1$, and $Z = \Sigma^{n-1}U$ for some object U. Choose $p: E \twoheadrightarrow B$ weakly equivalent to f and build the cube in (3.7). By (3.7), there is a cofibration $\Sigma(F \wedge Z) \rightarrowtail M' \longrightarrow M$ such that $M' \longrightarrow M$ is weakly equivalent to $C_{p'} \longrightarrow C_p \quad C_f$. Now, by induction, $\mathrm{Cl}(M') = \mathrm{Cl}(C_{p'}) \leq n$. Since $\Sigma(F \wedge \Sigma^{n-1}U) = \Sigma^n(F \wedge U)$ (see appendix), $\mathrm{Cl}(C_f) \leq n+1$.

In order to compare c and cat it is useful to know something about the behaviour of LS-invariants concerning Ganea-objects.

Proposition 3.9. $c(G^nB) \leq n$. *If c satisfies* (3.5)(i), *then also* $c(G^nB) \leq c(B)$. *If moreover* $c \geq \mathrm{cat}$, *then* $c(G^nB) = n$ *when* $n \leq \mathrm{cat}\, B$ *and* $\mathrm{cat}\, B \leq c(G^nB) \leq c(B)$ *when* $n \geq \mathrm{cat}\, B$.

Proof: We prove the first two statements simultaneously by induction. If $n = 0$, $G^n B \quad 0$. Hence, by LS1 and LS2, $c(G^n B) = 0$, in particular $c(G^n B) \leq c(B)$. For $n > 0$ consider a $n-1^{\text{st}}$Ganea fibration $F \longrightarrow G^{n-1}B \twoheadrightarrow B$. Then $G^n B \quad G^{n-1}B \vee_F CF$. By LS3, $c(G^n B) \leq c(G^{n-1}B) + 1 \leq n$. By (3.5)(i), $c(G^n B) \leq c(B)$.

In order to prove the third assertion let first $n \leq \operatorname{cat} B$. We know already $c(G^n B) \leq n$, and we have seen $c(G^i B) \leq c(G^{i-1}B)+1$. If $c(G^n B)$ was strictly less than n, $c(G^{\operatorname{cat} B} B)$ would hence also be strictly less than $\operatorname{cat} B$. But, by (2.4), $G^{\operatorname{cat} B} B$ dominates B [3], and hence, by [10|(4.6)], $\operatorname{cat} B \leq \operatorname{cat} G^{\operatorname{cat} B} B \leq c(G^{\operatorname{cat} B} B)$, a contradiction. Thus, $c(G^n B) = n$ when $n \leq \operatorname{cat} B$.

Let now $n \geq \operatorname{cat} B$. Then $G^n B$ dominates B. Thus, $\operatorname{cat} B \leq \operatorname{cat} G^n B \leq c(G^n B) \leq c(B)$.

The inequality $c \leq \operatorname{cat} + 1$ is based on the following

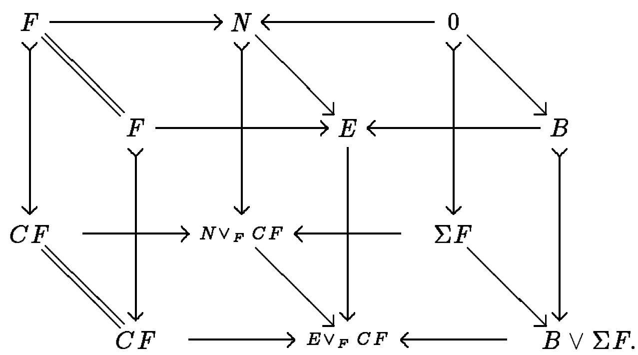
Theorem 3.10. *Assume that* $\operatorname{cat} B \leq n$. *Then there is an object* F *such that* $c(B \vee \Sigma F) \leq n$.

Proof: Without loss of generality we can assume that c satisfies (3.5)(i) and that B is cofibrant and 0-fibrant. Fix a n^{th} Ganea fibration $G^n B \twoheadrightarrow B$. Since $\operatorname{cat} B \leq n$, it has a section $s \colon B \longrightarrow G^n B$. Convert s into a fibration: $s \colon B \longrightarrow E \twoheadrightarrow G^n B$. Next build

$$\begin{array}{ccccc}
F & \longrightarrow & E & \twoheadrightarrow & G^n B \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow\!\!\downarrow \\
N & \longrightarrow & E & \twoheadrightarrow & B \\
\downarrow\!\!\downarrow & & & & \\
0 & & & &
\end{array}$$

where F and N are the respective fibres. Now consider

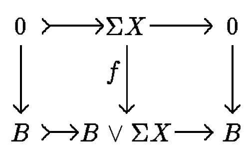
[3] An object X dominates an object Y if there is a map $X \longrightarrow Y$ in the homotopy category which admits a section.



Now, $c(B \vee \Sigma X) = c(E \vee_F CF) \leq$ (3.5)(i) $c(G^n B) \leq n$.

Corollary 3.11. $c \leq \mathrm{cat} + 1$.

Proof: Assume $\mathrm{cat}\, B \leq n$. Then there is an object X such that (for every 0-cofibrant ΣX) $c(B \vee \Sigma X) \leq n$. By composition, the right hand square in



is a homotopy push-out. Thus C_f B, and hence $c(B) = c(C_f) \leq$ (LS3) $c(B \vee \Sigma X) + 1 \leq n + 1$.

Remarks 3.12. (i) Combining (3.6), (3.8), and (3.11), we get $c \leq \mathrm{Cat} = \mathrm{Cl} \leq \mathrm{cat} + 1$.
(ii) The proof of (3.10) is completely analogous to the proof of Thm.1.1. in [3]. There it is shown that if a space X has category n, then there is a space Z such that $\mathrm{Cl}(X \vee \Sigma^n Z) \leq n$. The n-fold suspension is obtained as follows: What we called F in the proof of (3.10) is in fact $\Omega\Sigma^n(\wedge^{n+1}\Omega B)$. Thus, ΣF is a n-fold suspension. For spaces we can hence replace Σ in (3.10) by Σ^n. We leave the question open if this is also possible in arbitrary J-categories.

It is conjectured that cat and Cat coincide for rational spaces. This suggests the conjecture that there are J-categories in which cat equals Cat. As we remarked already in (2.7), this cannot hold for all J-categories. For the decision whether conjectures concerning the coincidence of LS-invariants hold in special J-categories or not it could be useful to have criteria of the following type:

Proposition 3.12. *For* $c \geq \mathrm{cat}$ *the following statements are equivalent:*
(a) $c = \mathrm{cat}$.
(b) *X dominates $Y \Rightarrow c(Y) \leq c(X)$.*
(c) *For a fibration $F \xrightarrow{i} E \twoheadrightarrow B$ with $\{i\} = 0$, $c(E) \leq c(B)$.*

<u>Proof</u>: (a) $\Rightarrow$ (b): [10|(4.6)].
(b) $\Rightarrow$ (a): Suppose $\mathrm{cat}\, B \leq n$. Then $G^n B$ dominates B. Hence $c(B) \leq c(G^n B) \leq n$.
(a) $\Rightarrow$ (c): [10|(4.19)].
(c) $\Rightarrow$ (a): Suppose $\mathrm{cat}\, B \leq n$. We have to show that $c(B) \leq n$. Without loss of generality we can assume that B is cofibrant and 0-fibrant. Fix a n^{th} Ganea fibration $G^n B \twoheadrightarrow B$. Since $\mathrm{cat}\, B \leq n$, it has a section $s: B \longrightarrow G^n B$. Convert s into a fibration: $s: B \longrightarrow E \twoheadrightarrow G^n B$. As in the proof of (3.10) build

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \twoheadrightarrow & G^n B \\ \downarrow & & \downarrow \mathrm{id} & & \downarrow\!\!\downarrow \\ N & \longrightarrow & E & \twoheadrightarrow & B \end{array}$$

where F and N are the fibres. Since N 0, $\{i\} = 0$. Thus, $c(B) = c(E) \leq c(G^n B) \leq n$.

Our last subject is to study the behaviour of LS-invariants under adjoint functors satisfying Quillen's conditions for the equivalence of homotopy categories. We will show that such functors preserve on one hand the class of LS-invariants and on the other hand the special LS-invariants cat and Cat.

Let $L: \mathcal{C} \rightleftarrows \mathcal{D}: R$ be a pair of adjoint functors between J-categories, L left-adjoint to R (i.e., there are natural bijections

$^{\flat}: \mathcal{M} \quad _{\mathcal{C}}(X, RB) \cong \mathcal{M} \quad _{\mathcal{D}}(LX, B): ^{\#}$), such that the following conditions hold:

(a) L carries cofibrations into cofibrations. R carries fibrations into fibrations.

(b) L carries weak equivalences between 0-cofibrant objects into weak equivalences. R carries weak equivalences between 0-fibrant objects into weak equivalences.

(c) Let X be a 0-cofibrant object in $\mathcal{C}$ and B be a 0-fibrant object in $\mathcal{D}$. Then a map $f: X \longrightarrow RB$ is a weak equivalence iff its adjoint $f^{\flat}: LX \longrightarrow B$ is a weak equivalence.

In [12] Quillen showed that L and R induce equivalences between the homotopy categories (at least if $\mathcal{C}$ and $\mathcal{D}$ are model categories).

Let $\mathcal{LS}(\mathcal{C})$, resp. $\mathcal{LS}(\mathcal{D})$, denote the class of LS-invariants defined on $\mathcal{O} \ \mathcal{C}$, resp. $\mathcal{O} \ \mathcal{D}$. For a LS-invariant $c \in \mathcal{LS}(\mathcal{C})$ define $R^*c: \mathcal{O} \ \mathcal{D} \longrightarrow \quad \cup \{\infty\}$ as follows: For $X \in \mathcal{O} \ \mathcal{D}$ choose a 0-fibrant model X^f and set $R^*c(X) := c(RX^f)$. As for another 0-fibrant model Y, $RX^f \quad RY$, this definition does not depend on the choice of the model. For a LS-invariant $c \in \mathcal{LS}(\mathcal{D})$ define $L^*c: \mathcal{O} \ \mathcal{C} \longrightarrow \quad \cup \{\infty\}$ similarly: For $X \in \mathcal{O} \ \mathcal{C}$ choose a 0-cofibrant model X^c and set $L^*c(X) := c(LX^c)$. As before, this definition does not depend on the choice of the model.

Proposition 3.13. *R^* and L^* are well-defined inverse bijections $R^*: \mathcal{LS}(\mathcal{C}) \cong \mathcal{LS}(\mathcal{D}): L^*$.*

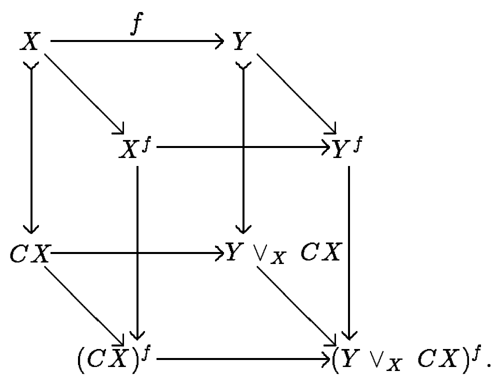
Proof: We have to prove
(a) $c \in \mathcal{LS}(\mathcal{C}) \Rightarrow R^*c \in \mathcal{LS}(\mathcal{D})$.
(b) $c \in \mathcal{LS}(\mathcal{D}) \Rightarrow L^*c \in \mathcal{LS}(\mathcal{C})$.
(c) $R^*L^* = \mathrm{id}_{\mathcal{LS}(\mathcal{D})}$.
(d) $L^*R^* = \mathrm{id}_{\mathcal{LS}(\mathcal{C})}$.

(a) Let $c \in \mathcal{LS}(\mathcal{C})$.

LS1: Let $X, Y \in \mathcal{D}$. $X \quad Y \Rightarrow X^f \quad Y^f \Rightarrow RX^f \quad RY^f \Rightarrow$ $R^*c(X) = c(RX^f) = c(RY^f) = R^*c(Y)$.

LS2: $R^*c(0) = c(R(0)) = c(0) = 0$.

LS3: Given $f\colon X \longrightarrow Y \in \mathcal{D}$, build



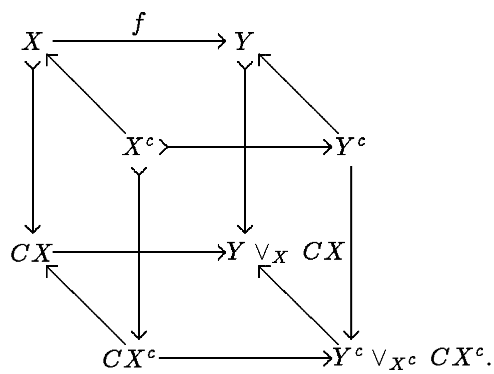
Then $X^f - Y^f - (Y \vee_X CX)^f - (CX)^f$ is a homotopy push-out. This is the case iff $RX^f - RY^f - R(Y \vee_X CX)^f - R(CX)^f$ is a homotopy push-out (see [10|(5.1)]). Thus, since $R(CX)^f \quad 0$, $R^*c(Y \vee_X CX) = c(R(Y \vee_X CX)^f) \leq c(RY^f) + 1 = R^*c(Y) + 1$.

(b) Let $c \in \mathcal{LS}(\mathcal{D})$.

LS1: Let $X, Y \in \mathcal{C}$. $X \quad Y \Rightarrow X^c \quad Y^c \Rightarrow LX^c \quad LY^c \Rightarrow$ $L^*c(X) = c(LX^c) = c(LY^c) = L^*c(Y)$.

LS2: $L^*c(0) = c(L(0)) = c(0) = 0$.

LS3: For $f\colon X \longrightarrow Y \in \mathcal{C}$ build



Then $LX^c - LY^c - L(Y^c \vee_{X^c} CX^c) - L(CX^c)$ is a (homotopy) push-out. Since $L(CX^c) = C(LX^c)$, $L^*c(Y \vee_X CX) = c(L(Y^c \vee_{X^c} CX^c)) \leq c(LY^c) + 1 = L^*c(Y) + 1$.

(c) Let $c \in \mathcal{LS}(\mathcal{D})$ and $X \in \mathcal{D}$. Choose $X \longrightarrow X^f$ and $(RX^f)^c \longrightarrow RX^f$. Then $L(RX^f)^c \longrightarrow X^f$. Now, $R^*L^*c(X) = L^*c(RX^f) = c(L(RX^f)^c) = c(X^f) = c(X)$.

(d) Let $c \in \mathcal{LS}(\mathcal{C})$ and $X \in \mathcal{C}$. Choose $X^c \longrightarrow X$ and $LX^c \longrightarrow (LX^c)^f$. Then $X^c \longrightarrow R(LX^c)^f$. Now, $L^*R^*c(X) = R^*c(LX^c) = c(R(LX^c)^f) = c(X^c) = c(X)$.

Without proof we remark that the bijections L^* and R^* restrict to bijections on the classes of LS-invariants satisfying (3.5)(i).

There are also bijections L^* and R^* between the classes of LS-invariants in the sense of (3.1). For a LS-invariant $c\colon \mathcal{M}\ \ \mathcal{C} \longrightarrow\ \ \cup \{\infty\}$ and $f \in \mathcal{M}\ \ \mathcal{D}$, $R^*c(f)$ is defined to be $c(Rp)$ where p is a morphism between 0-fibrant objects such that $f\ \ p$. L^* is defined similarly. As LS-invariants in the sense of (3.1) are determined by their behaviour on the 0-level, for LS-invariants c defined on $\mathcal{M}\ \ \mathcal{C}$ and d defined on $\mathcal{M}\ \ \mathcal{D}$, $R^*c = d$ (resp. $L^*d = c$) iff $R^*c = 0 \Longleftrightarrow d = 0$ (resp. $L^*d = 0 \Longleftrightarrow c = 0$). Using this criterion it is easy to see that $R^*\,\mathrm{cat} = \mathrm{cat}$, $L^*\,\mathrm{cat} = \mathrm{cat}$, $R^*\,\mathrm{Cat} = \mathrm{Cat}$, and $L^*\,\mathrm{Cat} = \mathrm{Cat}$. As the definitions of R^* and L^* for LS-invariants in the sense of (3.1) and in the sense of (3.4) are compatible with the passage to object-functions, we also have these equalities for R^* and L^* as defined before (3.13).

As we did not prove that R^* and L^* are well-defined bijections between the classes of LS-invariants in the sense of (3.1), we will prove directly that L^* and R^* as defined before (3.13) preserve cat and Cat.

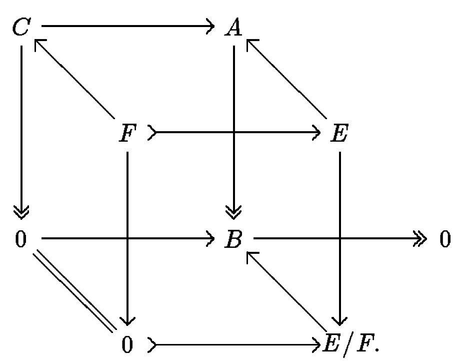
Proposition 3.14. (i) $R^* \operatorname{cat} = \operatorname{cat}$ *and* $L^* \operatorname{cat} = \operatorname{cat}$.
(ii) $R^* \operatorname{Cat} = \operatorname{Cat}$ *and* $L^* \operatorname{Cat} = \operatorname{Cat}$.

Proof: (i) [10|(5.7)].
(ii) We show inductively $R^* \operatorname{Cat}_{\mathcal{C}} X \leq n \iff \operatorname{Cat}_{\mathcal{D}} X \leq n$.

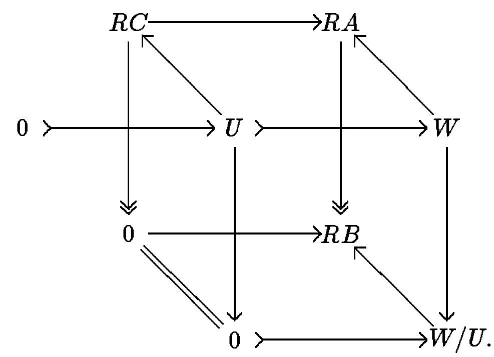
$n = 0$: $R^* \operatorname{Cat}_{\mathcal{C}} X = 0 \iff \operatorname{Cat}_{\mathcal{C}} RX^f = 0 \iff RX^f \quad 0 \iff X^f \quad 0 \iff X \quad 0 \iff \operatorname{Cat}_{\mathcal{C}} X = 0$.

$n > 0$: "$\Rightarrow$" Let $R^* \operatorname{Cat}_{\mathcal{C}} X \leq n$. Then $\operatorname{Cat}_{\mathcal{C}} RX^f \leq n$. Thus, there is a cofibration $F \rightarrowtail E \longrightarrow E/F \quad RX^f$ such that $\operatorname{Cat}_{\mathcal{C}} E \leq n-1$. We can assume that F is 0-cofibrant. Applying L we get a cofibration $LF \rightarrowtail LE \longrightarrow L(E/F) = LE/LF \quad X^f \quad X$. Now, $R^* \operatorname{Cat}_{\mathcal{C}} LE = \operatorname{Cat}_{\mathcal{C}} R(LE)^f = \operatorname{Cat}_{\mathcal{C}} E \leq n-1$. By induction, $\operatorname{Cat}_{\mathcal{D}} LE \leq n-1$. Thus, $\operatorname{Cat}_{\mathcal{D}} X \leq n$.

"$\Leftarrow$" Let $\operatorname{Cat}_{\mathcal{D}} X \leq n$. Then there is a cofibration $F \rightarrowtail E \longrightarrow E/F \quad X$ such that $\operatorname{Cat}_{\mathcal{D}} E \leq n-1$. Build



Then $RC - RA - RB - 0$ is a homotopy push-out. Build



Now, $\operatorname{Cat}_{\mathcal{D}} A \leq n-1$. Thus, by induction, $\operatorname{Cat}_{\mathcal{C}} W = \operatorname{Cat}_{\mathcal{C}} RA = R^* \operatorname{Cat}_{\mathcal{C}} A \leq n-1$. It follows that $R^* \operatorname{Cat}_{\mathcal{C}} X = R^* \operatorname{Cat}_{\mathcal{C}} B = \operatorname{Cat}_{\mathcal{C}} RB \leq n$.

We have proved that $\operatorname{Cat}_{\mathcal{D}} = R^* \operatorname{Cat}_{\mathcal{C}}$. Finally $L^* \operatorname{Cat}_{\mathcal{D}} = L^* R^* \operatorname{Cat}_{\mathcal{C}} = \operatorname{Cat}_{\mathcal{C}}$.

Appendix: Joins, smash-products, and suspensions

For topological spaces A and B, we have (up to homotopy) the equalities $A * B = \Sigma(A \wedge B) = A \wedge \Sigma B$ and $\Sigma^n(A \wedge B) = A \wedge \Sigma^n B$. These equalities also hold in arbitrary J-categories. As we used this in (3.7) and (3.8), we will give a proof here. We first recall some definitions.

Let X be any object of $\mathcal{C}$. A *cone* CX over X is given by a factorization $X \rightarrowtail CX \longrightarrow 0$. The *homotopy cofibre* C_f of a map $f\colon X \longrightarrow Y$ is up to weak equivalence defined by the push-out

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \text{p.o.} & \downarrow \\ CX & \longrightarrow & C_f. \end{array}$$

Note that weakly equivalent maps have weakly equivalent homotopy cofibres.

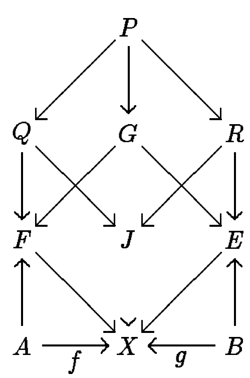
An object which is weakly equivalent to "the" homotopy cofibre of $X \longrightarrow 0$ is called a *suspension* of X and is denoted ΣX. Clearly, all suspensions of X are weakly equivalent. Moreover, if $X \quad Y$, an object is a suspension of X iff it is a suspension of Y.

If X is 0-cofibrant, a *cylinder* IX of X is given by a factorization $\mathrm{id} + \mathrm{id}\colon X \vee X \rightarrowtail IX \longrightarrow X$. If X is 0-cofibrant, a suspension is given by the push-out

$$\begin{array}{ccc} X \vee X & \longrightarrow & 0 \\ \downarrow & \text{p.o.} & \downarrow \\ IX & \longrightarrow & \Sigma X. \end{array}$$

Let A and B be objects of $\mathcal{C}$. An object is called a *smash-product* of A and B if it is weakly equivalent to the homotopy cofibre of $(\mathrm{id}, 0) + (0, \mathrm{id})\colon A^f \vee B^c \longrightarrow A^f \times B^c$ where A^f is a 0-fibrant model of A and B^c is a 0-cofibrant model of B. All smash-products of A and B are weakly equivalent and are denoted $A \wedge B$. If $A \quad A'$ and $B \quad B'$, then a smash-product of A and B is also a smash-product of A' and B' and vice versa. Moreover, a smash product of A and B is also a smash product of B and A (and vice versa).

A commutative diagram



is called a *join-diagram* of f and g if the following conditions hold:
(a) $E \longrightarrow X$ or $F \longrightarrow X$ is a fibration.
(b) $P \longrightarrow Q$ or $P \longrightarrow R$ is a cofibration.
(c) $G - E - X - F$ is a pull-back and $P - R - J - Q$ is a push-out.

An object is called a *join* of A and B if it is weakly equivalent to the object J in a join diagram of $A \longrightarrow 0$ and $B \longrightarrow 0$. All joins of A and B are weakly equivalent and are denoted $A * B$. If A' A and B' B, then a join of A and B is also a join of A' and B'. Moreover, a join of A and B is also a join of B and A.

The proof of the following lemma is left to the reader.

Lemma A.1. *Let A be 0-fibrant and B be 0-cofibrant. Let $P \underset{\psi}{\longrightarrow} A \times B$ be a 0-cofibrant model and $P \vee P \rightarrowtail^{\iota} IP \xrightarrow{\rho} P$ be a cylinder. Then there is a push-out*

$$\begin{array}{ccc} P \vee P & \overset{\iota}{\rightarrowtail} & IP \\ {\scriptstyle \mathrm{pr}_A \psi \vee \mathrm{pr}_B \psi} \downarrow & \text{p.o.} & \downarrow \\ A \vee B & \rightarrowtail & A * B. \end{array}$$

Proposition A.2. $A * B = \Sigma(A \wedge B) = A \wedge \Sigma B$.

<u>Proof</u>: The geometrical idea for the proof is that the objects in question all result from projecting the subspace

$$U = (A \vee B) \times I \cup A \times B \times \{0,1\} \Big/ \{*\} \times \{*\} \times I$$

of $A \times B \times I \Big/ \{*\} \times \{*\} \times I$ to a contractible space.

We may assume that A is 0-fibrant and that B is 0-cofibrant. Unfortunately, we can neither assume that A nor that $A \times B$ is 0-cofibrant. As we want to work with cylinders and these are only defined for 0-cofibrant objects, we hence have to choose $0 \rightarrowtail A' \underset{\alpha}{\longrightarrow} A$ and

$$\begin{array}{ccc} A' \vee B & \longrightarrow & A \vee B \\ \big\downarrow\!\!\!\!\curlyvee & & \big\downarrow \\ P & \xrightarrow[\psi]{} & A \times B. \end{array}$$

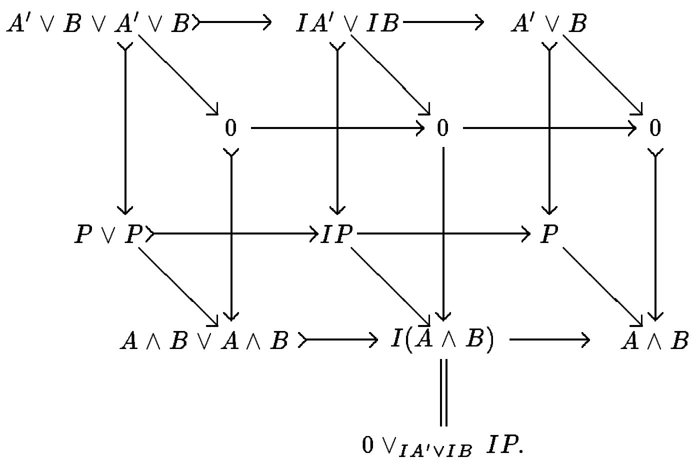
Note that we have a push-out

$$\begin{array}{ccc} A' \vee B & \longrightarrow & 0 \\ \big\downarrow\!\!\!\!\curlyvee & \text{p.o.} & \big\downarrow\!\!\!\!\curlyvee \\ P & \longrightarrow & A \wedge B. \end{array}$$

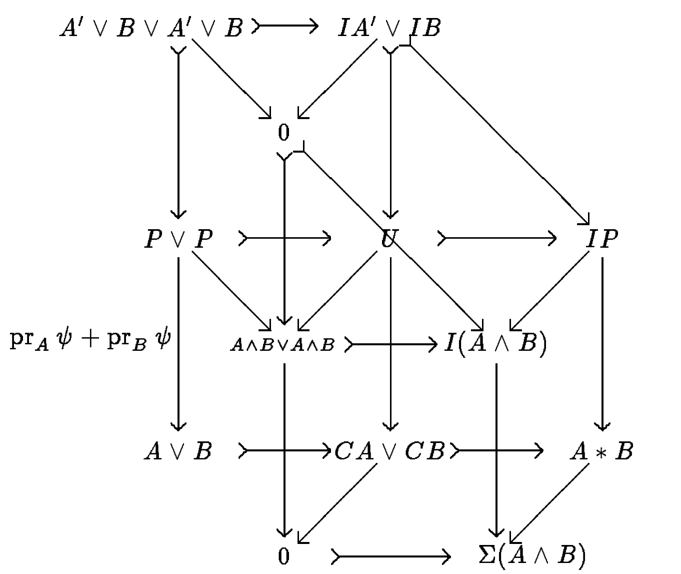
Next choose cylinders $A' \vee A' \overset{\iota_{A'}}{\rightarrowtail} IA' \xrightarrow[\rho_{A'}]{} A'$ and $B \vee B \overset{\iota_B}{\rightarrowtail} IB \xrightarrow[\rho_B]{} B$. With these we define U and a cylinder of P:

$$\begin{array}{ccccc} A' \vee B \vee A' \vee B & \rightarrowtail & P \vee P & & \\ \big\downarrow\!\!\!\!\curlyvee & \text{p.o.} & \big\downarrow\!\!\!\!\curlyvee & \overset{\iota_P}{\searrow} & \\ IA' \vee IB & \rightarrowtail & U & \rightarrowtail & IP \\ \big\downarrow & & \big\downarrow\!\!\!\!\curlyvee & \underset{\rho_P}{\swarrow} & \\ A' \vee B & \rightarrowtail & P. & & \end{array}$$

Automatically, we obtain a cylinder of $A \wedge B$:



As mentioned above we want to show that $A * B$, $\Sigma(A \wedge B)$, and $A \wedge \Sigma B$ result from projecting U to contractible objects. For $A * B$ and $\Sigma(A \wedge B)$ we build



where all squares except the left and the right face of the cube are push-outs.

It remains to show that also $A \wedge \Sigma B$ is obtained from IP by projecting U to a contractible object, i.e. that there is a push-out

$$\begin{array}{ccc} U & \rightarrowtail & IP \\ \downarrow & \text{p.o.} & \downarrow \\ 0 & \rightarrowtail & A \wedge \Sigma B. \end{array}$$

For this it suffices to show that there is a push-out

$$\begin{array}{ccc} U & \rightarrowtail & IP \\ \downarrow & \text{p.o.} & \downarrow \\ X & \underset{f}{\rightarrowtail} & Y \end{array}$$

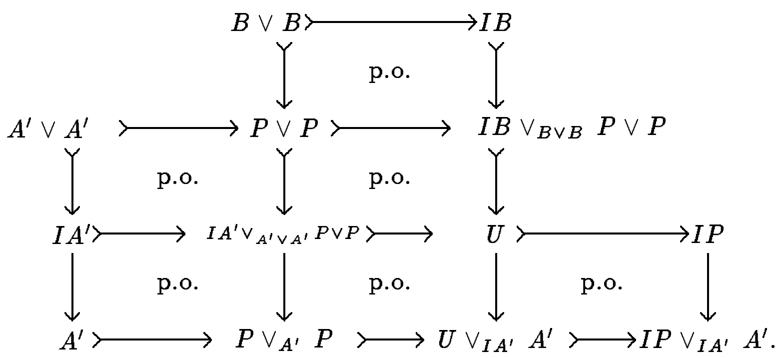
and a 0-cofibrant ΣB such that f is weakly equivalent to $(\mathrm{id}, 0) + (0, \mathrm{id})\colon A \vee \Sigma B \longrightarrow A \times \Sigma B$.

We will show that $U \rightarrowtail IP$ is weaskly equivalent to $\mu := (\mathrm{id} \times \iota_B) + (0, \mathrm{id})\colon A \times (B \vee B) \vee_{B \vee B} IB \longrightarrow A \times IB$. Then we will find a push-out $U - IP - Y - X$ because we have the following commutative diagram:

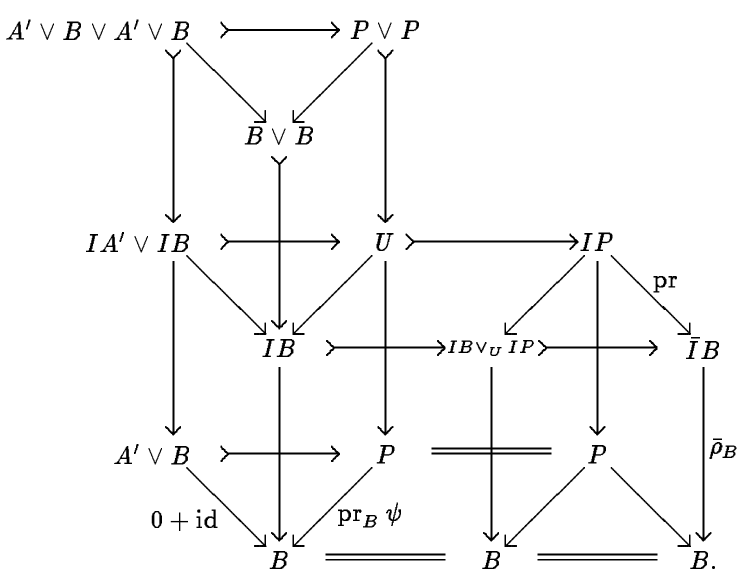
$$\begin{array}{ccccc} B \vee B & \overset{\iota_B}{\rightarrowtail} & IB & \overset{(0,\mathrm{id})}{\searrow} & \\ {\scriptstyle (0,\mathrm{id})}\downarrow & \text{(h.)p.o.} & \downarrow & & \\ A \times (B \vee B) & \rightarrowtail & A \times (B \vee B) \vee_{B \vee B} IB & \overset{\mu}{\longrightarrow} & A \times IB \\ {\scriptstyle \mathrm{pr}_A}\downarrow & \text{(h.)p.o.} & \downarrow & \text{h.p.o.} & \downarrow \\ A & \rightarrowtail & A \vee \Sigma B & \longrightarrow & A \times \Sigma B. \end{array}$$

Note that $A \times (B \vee B) - A \times IB - A \times \Sigma B - A$ is a homotopy push-out because we are working in a J-category.

In order to show that $U \rightarrowtail IP$ and μ are weakly equivalent we first project IA' in U and IP to A'. This is done in



The map $U \vee_{IA'} A' \rightarrowtail IP \vee_{IA'} A'$ is identical with the map $\nu := \iota_P \vee_{\iota_{A'}} \mathrm{id}_{A'} + j\colon (P \vee P \vee_{A' \vee A'} A') \vee_{B \vee B} IB \longrightarrow IP \vee_{IA'} A'$ where j is the composition on the right side of the above diagram. We are done when we can show that μ and ν are weakly equivalent. Unfortunately, we cannot construct a weak equivalence $\nu \longrightarrow \mu$ without further ado because we cannot project IP onto IB. In order to get such a projection we construct a second cylinder of B:



We will denote the map between the cylinders of B by $1\colon IB \rightarrowtail \bar{I}B$.

We will construct a weak equivalence $\nu \longrightarrow (\mathrm{id}_A \times 1)\mu$. This suffices because $\mathrm{id}_A \times 1$ is a weak equivalence. First we construct a weak equivalence between the sources of ν and $(\mathrm{id}_A \times 1)\mu$. Let $i_1, i_2\colon B \longrightarrow B \vee B$ be the inclusions. As we are working in a J-category, the following diagram is a homotopy push-out:

$$\begin{array}{ccc} A & \xrightarrow{(\mathrm{id},0)} & A \times B \\ {\scriptstyle (\mathrm{id},0)}\big\downarrow & \text{h.p.o.} & \big\downarrow{\scriptstyle \mathrm{id} \times i_2} \\ A \times B & \xrightarrow[\mathrm{id} \times i_1]{} & A \times (B \vee B). \end{array}$$

Thus, $\varphi := ((\mathrm{id}_A \times i_1)\psi + (\mathrm{id}_A \times i_2)\psi) + (\alpha, 0)\colon P \vee P \vee_{A' \vee A'} A' \longrightarrow A \times (B \vee B)$ is a weak equivalence. As φ followed by the composition $B \vee B \rightarrowtail P \vee P \rightarrowtail P \vee P \vee_{A' \vee A'} A'$ is $(0, \mathrm{id})\colon B \vee B \longrightarrow A \times (B \vee B)$, we get a weak equivalence $\varphi \vee_{\mathrm{id}} \mathrm{id}\colon (P \vee P \vee_{A' \vee A'} A') \vee_{B \vee B} IB \longrightarrow A \times (B \vee B) \vee_{B \vee B} IB$.

The map $(\mathrm{pr}_A \psi \rho_P, \mathrm{pr}) + (\alpha, 0) : IP \vee_{IA'} A' \longrightarrow A \times \bar{I}B$ between the targets of ν and $(\mathrm{id}_A \times 1)\mu$ is well-defined. It is a weak equivalence because we have the following commutative diagram:

$$\begin{array}{ccc} IP \vee_{IA'} A' & \xrightarrow{\rho_P \vee_{\rho_{A'}} \mathrm{id}} & P \vee_{A'} A' = P \\ {\scriptstyle (\mathrm{pr}_A \psi \rho_P, \mathrm{pr}) + (\alpha, 0)}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\ A \times \bar{I}B & \xrightarrow[\mathrm{id} \times \bar{\rho}_B]{} & A \times B. \end{array}$$

To end one has to check that

$$\begin{array}{ccc} (P \vee P \vee_{A' \vee A'} A') \vee_{B \vee B} IB & \xrightarrow{\nu} & IP \vee_{IA'} A' \\ {\scriptstyle \varphi \vee_{\mathrm{id}} \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{pr}_A \psi \rho_P, \mathrm{pr}) + (\alpha, 0)} \\ A \times (B \vee B) \vee_{B \vee B} IB & \xrightarrow[(\mathrm{id} \times 1)\mu]{} & A \times \bar{I}B \end{array}$$

is commutative. This is a straight forward computation which we leave to the reader.

The other formula is an immediate consequence.

Corollary A.3. $\Sigma^n(A \wedge B) = A \wedge \Sigma^n B$.

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