

Demonstrações simples do teorema forte da eliminação do corte

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Goal

1. logic: intuitionistic implication \supset

$$2. SCE = SN + \epsilon$$

3. More precisely:

$$SN \text{ for } \lambda^G = (\text{properties of } \lambda) + \epsilon$$

where

$$\lambda^G = \text{neat } \lambda\text{-calculus for sequent calculus}$$

$$\epsilon = \text{small combinatorial overhead}$$

4. Plan:

(a) advertise λ^G

(b) indicate properties of λ

(c) do ϵ

Preliminaries (1)

Types $A, B, C ::= a \mid A \supset B$

Terms $M, N ::= x \mid \lambda x.M \mid MN \mid \langle N/x \rangle M$

Sequents $\Gamma \vdash M : A$

Typing / Inference rules

$$\frac{}{\Gamma, x : A \vdash x : A} \textit{Axiom}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \supset B} \textit{Intro}$$

$$\frac{\Gamma \vdash M : A \supset B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \textit{Elim}$$

$$\frac{\Gamma \vdash N : A \quad \Gamma, x : A \vdash M : B}{\Gamma \vdash \langle N/x \rangle M : B} \textit{Subst}$$

Preliminaries (2)

Reduction rules

$$(\beta) \quad (\lambda x.M)N \rightarrow \langle N/x \rangle M$$

$$(\sigma) \quad \langle N/x \rangle M \rightarrow [N/x]M$$

$$\text{N.B. } [N/x]\langle N'/y \rangle M = \langle [N/x]N'/y \rangle [N/x]M$$

Logical reading (normalisation)

$$\frac{\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : \mathbf{A} \supset \mathbf{B}} I. \quad \frac{\Gamma \vdash N : A}{\Gamma \vdash (\lambda x.M)N : B} E.}{\Gamma \vdash (\lambda x.M)N : B} \mathcal{D} \rightarrow \frac{\frac{\Gamma \vdash N : \mathbf{A} \quad \Gamma, x : \mathbf{A} \vdash M : B}{\Gamma \vdash \langle N/x \rangle M : B} S.}{\Gamma \vdash \langle N/x \rangle M : B} \mathcal{D}' E.$$

$$\frac{\frac{\Gamma \vdash N : \mathbf{A} \quad \Gamma, x : \mathbf{A} \vdash M : B}{\Gamma \vdash \langle N/x \rangle M : B} S.}{\Gamma \vdash [N/x]M : B} \mathcal{D}' \rightarrow \frac{\Gamma \vdash [N/x]M : B}{\Gamma \vdash [N/x]M : B} \mathcal{D}''$$

where $\mathcal{D}'' = \text{right-perm}(\mathcal{D}', x, \mathcal{D})$.

Sequent calculus/ λ^G -calculus (1)

Expressions

(Terms) $t, u, v ::= x \mid \lambda x.t \mid tk$
(Contexts) $k ::= (x)v \mid u :: k$

Sequents $\Gamma \vdash t : A$ and $\Gamma; A \vdash k : B$

Typing / Inference rules

$\frac{}{\Gamma, x : A \vdash x : A}$ *Axiom* $\frac{\Gamma, x : A \vdash v : B}{\Gamma; A \vdash (x)v : B}$ *Selection*

$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \supset B}$ *Right*

$\frac{\Gamma \vdash u : A \quad \Gamma; B \vdash k : C}{\Gamma; A \supset B \vdash u :: k : C}$ *Left*

$\frac{\Gamma \vdash t : A \quad \Gamma; A \vdash k : B}{\Gamma \vdash tk : B}$ *Cut*

Sequent calculus/ λ^G -calculus (2)

Cuts: $tk = t(u_1 :: \dots :: u_m(x)v)$, $m \geq 0$

Particular cases:

$m = 1$	$t(u :: (x)v)$	generalized application
$m = 0$	$t(x)v$	“suspended” substitution
$v = x$	$t(u_1 :: \dots :: u_m(x)x)$	multiary application

N.B. $t(x)v$ is written $\langle t/x \rangle v$

Reduction rules

(β)	$(\lambda x.t)(u :: k)$	\rightarrow	$\langle u/x \rangle(tk)$
(σ)	$\langle t/x \rangle v$	\rightarrow	$subst(t, x, v)$
(π)	$(tk)k'$	\rightarrow	$t\ append(k, k')$

Logical reading (cut elimination)

(β)	key step
(σ)	right permutation
(π)	left permutation

Properties of λ

1. Strong normalisation

$$M \text{ typable} \Rightarrow M \in SN_\beta.$$

2. Fundamental lemma of perpetuality

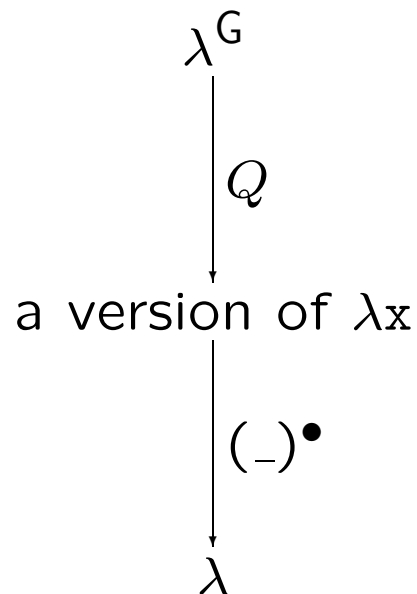
$$\left. \begin{array}{l} (\lambda x.M)N \notin SN_\beta \\ x \notin FV(M) \Rightarrow N \in SN_\beta \end{array} \right\} \Rightarrow [N/x]M \notin SN_\beta$$

Stronger version of 2.

$$x \in FV(M) \Rightarrow \|(\lambda x.M)N\|_\beta \leq \|[N/x]M\|_\beta + 1$$

$$x \notin FV(M) \Rightarrow \|(\lambda x.M)N\|_\beta \leq \|M\|_\beta + \|N\|_\beta + 1$$

Doing ϵ (1)



First step: reduce SN for λ^G to SN for a version of λ_x , with help of Q

Second step: enrich λ

Third step: prove SN for a version of λ_x , with help of $(-)^{\bullet}$

Doing ϵ (2)

First step: reduce SN for λ^G to SN for a version of λ_x

1. Recall the terms of λ_x :

$$M, N, P ::= x \mid \lambda x.M \mid MN \mid \langle N/x \rangle M$$

2. Equip this set with

$$\begin{aligned} (\beta) \quad & (\lambda x.M)N \rightarrow \langle N/x \rangle M \\ (\sigma) \quad & \langle N/x \rangle M \rightarrow [N/x]M \\ (\pi_1) \quad & (\langle P/x \rangle M)N \rightarrow \langle P/x \rangle (MN) \\ (\pi_2) \quad & \langle \langle P/y \rangle N/x \rangle M \rightarrow \langle P/y \rangle \langle N/x \rangle M \end{aligned}$$

and put $\pi = \pi_1 \cup \pi_2$

3. Define $Q : \lambda^G \rightarrow \lambda_x$ as follows:

$$t(u_1 :: \dots :: u_m(x)v) \mapsto \langle MN_1 \dots N_m/x \rangle P$$

4. SN for $\lambda_x \Rightarrow$ SN for λ^G

Doing ϵ (3)

Second step: enrich λ

1. Equip the set of λ -terms with

$$(\pi_1) \quad (\lambda x.M)NP \rightarrow (\lambda x.MP)N$$

$$(\pi_2) \quad M((\lambda y.P)N) \rightarrow (\lambda y.MP)N$$

and put $\pi = \pi_1 \cup \pi_2$

2. In λ , \rightarrow_π terminates

3. In λ , \rightarrow_π does not increase $\|-\|_\beta$

(fund. lemma perpetuality used here)

Doing \in (4)

Third step: conclude

1. Define $(-)^{\bullet} : \lambda_{\mathbf{x}} \rightarrow \lambda$ by

$$(\langle N/x \rangle M)^{\bullet} = (\lambda x.M^{\bullet})N^{\bullet}$$

2. In $\lambda_{\mathbf{x}}$, $\rightarrow_{\beta\pi}$ terminates

3. For all $M \in \lambda_{\mathbf{x}}$, $M^{\bullet} \in SN_{\beta} \Rightarrow M \in SN_{\beta\pi\sigma}$

4. Corollary (SN for $\lambda_{\mathbf{x}}$):

$$\text{For all } M \in \lambda_{\mathbf{x}}, M \text{ typable} \Rightarrow M \in SN_{\beta\pi\sigma}$$

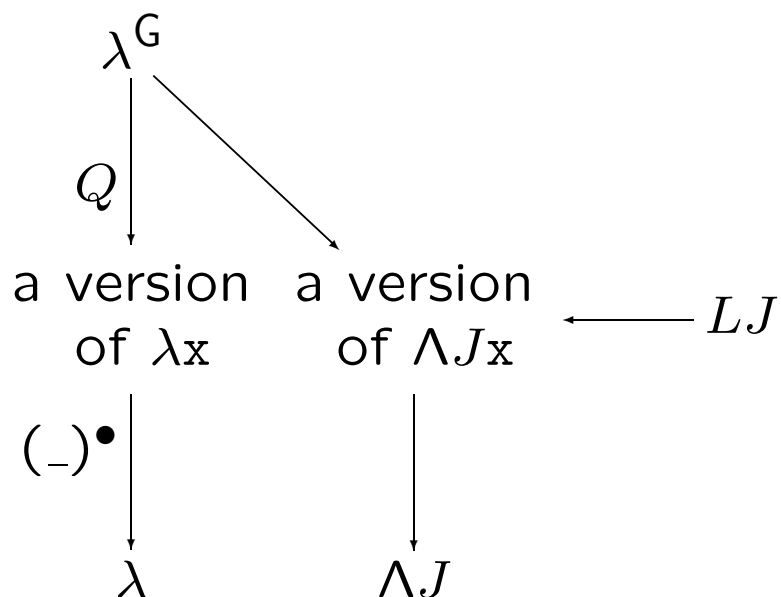
(SN for λ used here)

5. Theorem (SN for $\lambda^{\mathbf{G}}$):

$$\text{For all } t \in \lambda^{\mathbf{G}}, t \text{ typable} \Rightarrow t \in SN_{\beta\pi\sigma}$$

Consequences

1. SN for variants of λ with generalised application
2. SN for LJ equipped with various “protocols”



Conclusions

1. reuse a previous result instead of repeating a previous proof
2. substitution is suspended, but not explicitly executed
3. simple \Rightarrow stronger