Centro de Matemática da Universidade do Minho

SCIENTIST AS ALGORITHM

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Introduction: Newton-Bentley's correspondence

Newton-Bentley's correspondence led Newton to abandon the Stoic Cosmos of a finite distribution of matter in infinite space and to adopt the Atomist Universe in which matter is distributed throughout infinite space.

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Letter 1

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Letter 2

Newton had fully agreed with Bentley that gravity meant providence had created a universe of great precision.

The hypothesis of deriving the frame of the world by mechanical principles from matter evenly spread through the heavens being inconsistent with my system, I had considered it very little before your letters put me upon it, and therefore trouble you with a line or two more about it...

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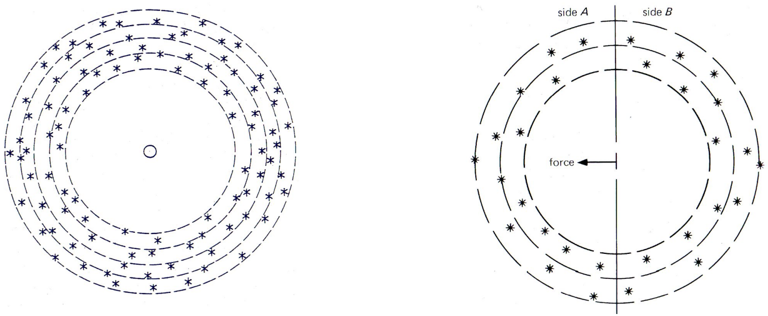

Figure: *Sensorium Dei* in Newton's metaphysics.

Given the initial conditions with infinite precision...

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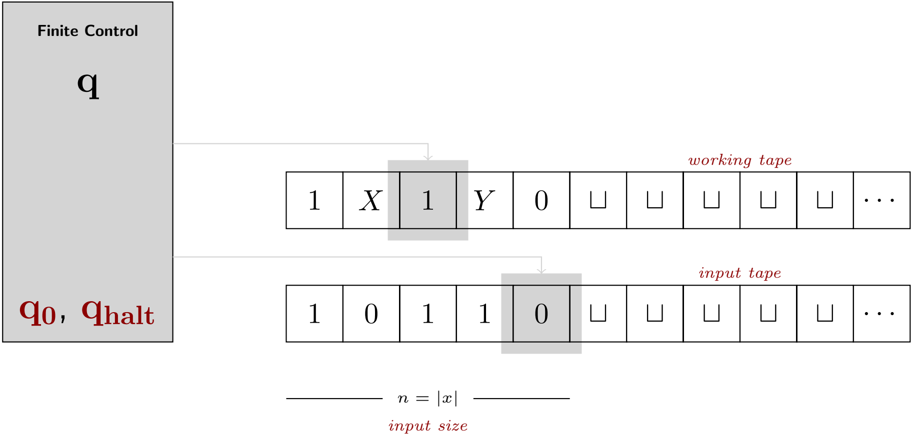
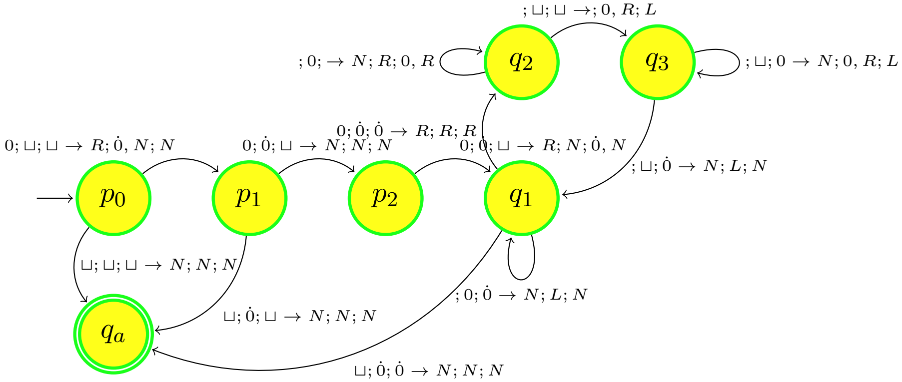
The Turing Machine

The Turing machine: How it works



Time constructible: The exponential



Sending pi away!

“

:$0 --> N$ $0 --> D$
: While 1
: rand$*2-1 --> X$
: rand$*2-1 --> Y$
: If $sqrt(x^2+y^2) =< 1$
: $N+1 --> N$
: End
: $D+1 --> D$
: Disp $(N/D*4)$
:End”

Poe, E.

Near a Raven

Midnights so dreary, tired and weary.
Silently pondering volumes extolling all by-now obsolete lore.
During my rather long nap – the weirdest tap!
An ominous vibrating sound disturbing my chamber's antedoor.
"This", I whispered quietly, "I ignore".

Perfectly, the intellect remembers: the ghostly fires, a glittering ember.
Inflamed by lightning's outbursts, windows cast penumbras upon this floor.
Sorrowful, as one mistreated, unhappy thoughts I heeded:
That inimitable lesson in elegance – Lenore –
Is delighting, exciting... nevermore.

(Mike Keith, 1995)

Specifying a Turing machine

Turing machine with $k > 2$ tapes, with an output tape; dynamic map

$\delta : Q \times \Gamma^{k-1} \rightarrow Q \times \Gamma^{k-1} \times \{L, N, R\}^k$

Example (Iterating the Collatz function: TM without output tape)

input n;

while $n \neq 1$ do if $even(n)$ then $n := n/2$ else $n := 3n + 1$

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Semantics of an algorithm

Computable functions, (total) recursive functions, $\mathcal{R}$

Each Turing machine $\mathcal{M}$ (with an output tape) computes a function $f_{\mathcal{M}} : \{0,1\}^\star \to \{0,1\}^\star$ (in other words, a function $f_{\mathcal{M}} : \mathbb{N} \to \mathbb{N}$); if the machine does not halt on an input x, then it is said that $f_{\mathcal{M}}$ is not defined at x.

Bijection between $\mathbb{N}$ and $\{0,1\}^\star$

Take a binary word, e.g., 0101, affix a leftmost 1 (10101), and read it in binary having subtracted 1 (20).
Take any number in decimal, e.g., 20, add 1, write it in binary (10101), remove the leftmost 1, and read the result as a binary word (0101).

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Theorem

There is a countable infinite number of computable functions, but an uncountable number of non-computable functions. (I.e., most of the functions $f : \{0,1\}^\star \to \{0,1\}^\star$ are non-algorithmic.)

Theorem (Universal Turing machine)

There exists an universal Turing machine $\mathcal{U}$ that receives as input $\langle \mathcal{M}, x \rangle$, the binary code of a Turing machine $\mathcal{M}$ and a binary word x, such that, for every such $\mathcal{M}$ and for every such x, it simulates $\mathcal{M}$ on input x:

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Growing rate and the halting problem

Theorem (Busy Beaver)

The Turing machine can not compute functions with arbitrary growing rate. There is a well-known established limit of growing rate for computable functions.

Theorem (The halting problem)

The halting function, namely the function

$$f(\langle y, x \rangle) = \begin{cases} 1 & \textit{if } \mathcal{M}_y \textit{ halts on } x \\ 0 & \textit{otherwise} \end{cases}$$

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Collatz function yet again

Example (Iterating the Collatz function)

input n;

while $n \neq 1$ do if $even(n)$ then $n := n/2$ else $n = 3n + 1$

Sequences of numbers produced for different inputs

4, 2, 1 HALT

5, 16, 8, 4, 2, 1 HALT

7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 HALT

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Decidable and semidecidable sets

Definition (Decidable set)

A set A (e.g., a subset of $\{0,1\}^\star$) is said to be decidable if the (characteristic) function (of A)

$$\xi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is Turing computable.

Definition (Semidecidable set)

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Diophantine sets

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We say that a relation D is Diophantine if there exists a polynomial p with integer coefficients, such that,

$$\langle m_1, \ldots, m_k \rangle \in D \text{ iff } \exists x_1, \ldots, x_n \in \mathbb{N} \; [\; p(m_1, \ldots, m_k, x_1, \ldots, x_n) = 0 \;]$$

Example (Composite numbers, divisible numbers, prime numbers, etc.)

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Example (Composite numbers, divisible numbers, prime numbers, etc.)

$$x \in Composite \quad \text{iff} \quad \exists y, z \in \mathbb{N} - \{0\} \; [\; (y+1)(z+1) - x = 0 \;]$$

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$$x|y \; and \; x < y \quad \text{iff} \quad \exists u, v \in \mathbb{N} - \{0\} \; [\; (xu - y)^2 + (y - x - v)^2 = 0 \;]$$

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A set is Diophantine if and only if it is semidecidable.

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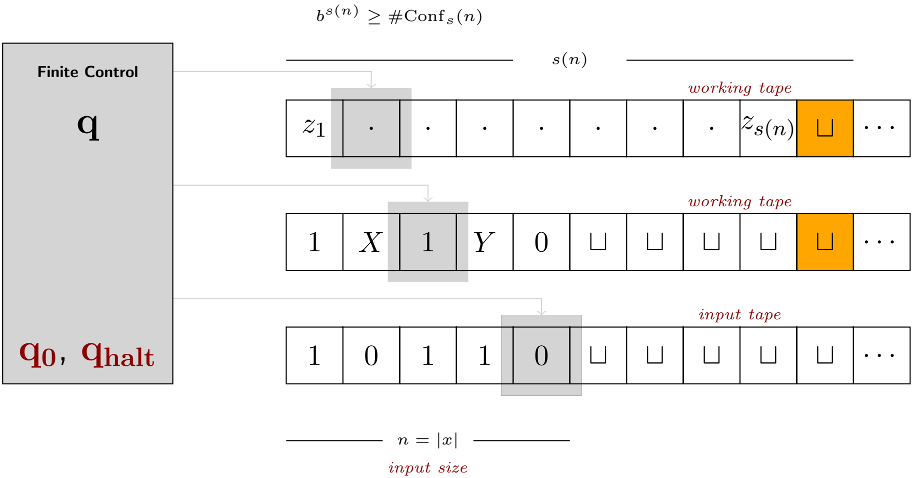
Halting Problem in bounded space



Halting Problem in bounded space

Theorem

The halting problem of Turing machines bounded in space is decidable.

Proof:

$$\begin{aligned}
\#\mathrm{Conf}_s(n) &= |Q| \times 3^{s(n)} \times s(n) \times n \\
&\in O(2^{O(s(n)} \times n) \\
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Conclusions of the section

1. The halting problem of Turing machines is decidable in bounded space;

2. The halting problem becomes interesting only when all the infinite tape is potentially used;

3. These are the important facts to take into consideration when we try to embed an infinite tape Turing machine into the physical space.

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Undecidability in Analysis and Physics

Undecidability in Analysis

Let $\mathcal{E}$ be a set of expressions denoting real, single valued, partially defined functions of one variable and let Φ be the set of functions denoted by expressions in $\mathcal{E}$.

Assumptions

1. *Φ contains the identity function, the rational numbers, and is closed under addition, subtraction, multiplication, and composition;*
2. *If $A, B \in \mathcal{E}$, then there is an effective procedure for finding expressions in $\mathcal{E}$ to denote $A(x) \pm B(x)$, $A(x) \times B(x)$, $A(B(x))$.*

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2. *If* $A, B \in \mathcal{E}$*, then there is an effective procedure for finding expressions in* $\mathcal{E}$ *to denote* $A(x) \pm B(x)$, $A(x) \times B(x)$, $A(B(x))$.

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The two big problems

1. The identity problem for $(\mathcal{E}, \Phi)$ is the problem of deciding, given $A \in \mathcal{E}$, whether $A(x) \equiv 0$.
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1. Φ *contains* π *and the real-valued function* $\sin(x)$;
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1. *If Φ satisfies condition 1, then the problem* given an expression $A \in \mathcal{E}$, decide if there is a real number x such that $A(x) < 0$ *is undecidable;*
2. *If Φ satisfies conditions 1 and 2, then the identity problem for $(\mathcal{E}, \Phi)$ is undecidable;*
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$\mathcal{E}$ is the smallest class of expressions obtained by iteration of addition, subtraction, multiplication, and composition, starting with x, e^x, $\sin(x)$ and $|x|$, and expressions for the rational numbers;

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The analytic machinery

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\begin{aligned}
h(x) &= x\sin(x) \\
g(x) &= x\sin(x^3) \\
x_1 &= h(x) \\
x_2 &= h \circ g(x) \\
x_3 &= h \circ g \circ g(x) \\
&\cdots \\
x_{n-1} &= h \circ \overbrace{g \circ \ldots \circ g}^{n-2}(x) \\
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Theorem (Richardson, 1968)

There is a elementary function of two variables, $G(m, x)$, such that, as m varies over $\mathbb{N}$, there is no algorithm for deciding whether is a real number x such that $G(m, x) \leq 1/2$.

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Theorem (Richardson [Ric68])

If Φ contains the identity function, the rational numbers, π, the real-valued functions of expressions $|x|$ and $\sin(x)$, and is closed under addition, subtraction, multiplication, and composition, then the identity problem for $(\mathcal{E}, \Phi)$ is undecidable.

Proof: Take $B(m,x) = |G(m,x) - 1| - (G(m,x) - 1)$. We have that $\exists x\ G(m,x) < 1$ if and only if $B(m,x) \not\equiv 0$. □

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Proof: If such integration problem were solvable, we would be able to decide, for each $m \in \mathbb{N}$, whether there were a function $f \in \Phi$ so that

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Undecidability in Physics

Are there general methods to test for the integrability of a given Hamiltonian? The answer, for the moment, is no. We can turn the question around, however, and ask if methods can be found to construct potentials that give rise to integrable Hamiltonians. The answer is that a method exists, at least for restricted class of problems, but the method becomes rapidly very tedious as the forms allowed for the integrals of the motion are expanded. *(A. J. Lichtenberg and M. A. Liberman,* Regular and Stochastic Motion.*)*

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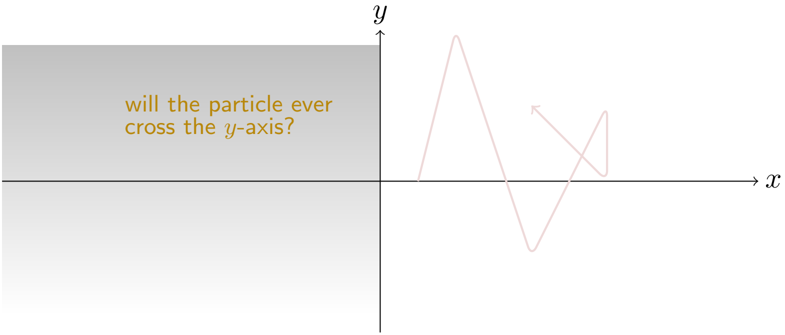

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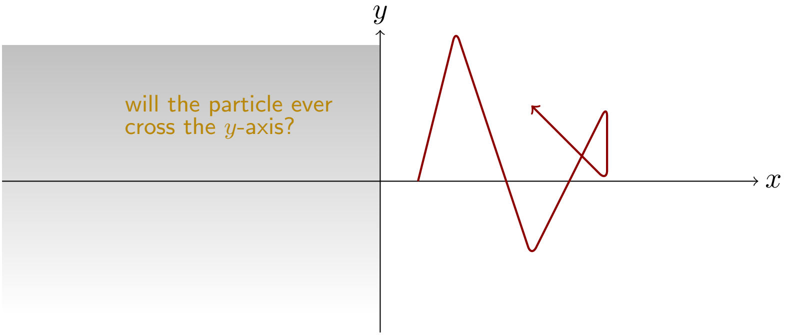

Motion in the plane

Proof: Take $x_m(t) = G(m,t) - 1$. There is no general decision procedure to check whether one has, given an arbitrary $m \in \mathbb{N}$, $x_m(t) < 0$ for some t. Take $m(t) = \langle x_m(t), \frac{1}{2}gt^2 \rangle$. □

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Off Infinite in Finite Time

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Definition (Singularity)

A singularity is a time value $t = t^\star$ where analytic continuation of the solution fails. It requires a distance $r_{ij}(t)$ to become arbitrarily small as $t \to t^\star$.

Example (Example and conjecture)

E.g., a collision is a singularity. But are all singularities collisions? Problem raised in the turn XIX/XX by Painlevé and Zeipel.

The way to the solution was provided by Sundman, Wintner, McGehee, Gerver, Saari, Xia.

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Non-collision singularity

Let

- $r_{\min}(t) = \min_{i \neq j} r_{ij}(t)$

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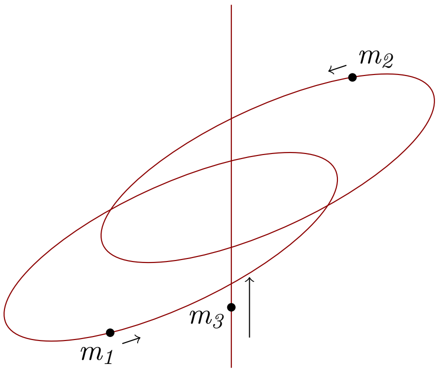

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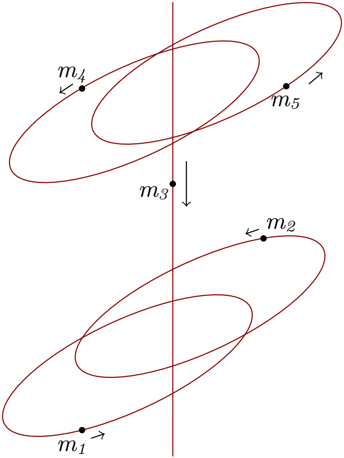

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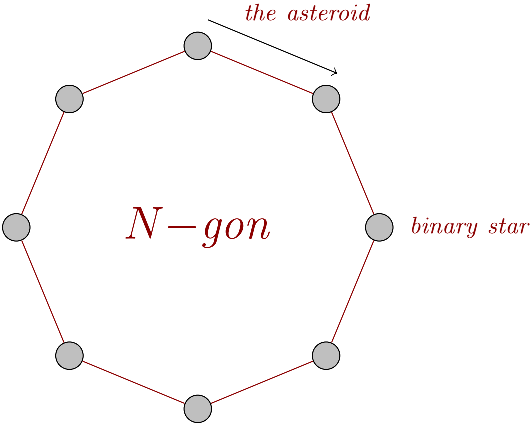

Topology

Theorem (Uncountably many topologies, Warren Smith [Smi06])

N point masses in the plane, for some finite fixed value of N, whose initial positions, masses, and velocities lie inside a cube in $\mathbb{R}^{7n}$, can describe an uncountably infinite number of topological distinct trajectories in 1 second. In contrast, a Turing machine simulator can only output one of a finite number of possible outputs, in a finite timespan. The initial location and velocities of the bodies required to force a future trajectory of described topological type, are computable real numbers.

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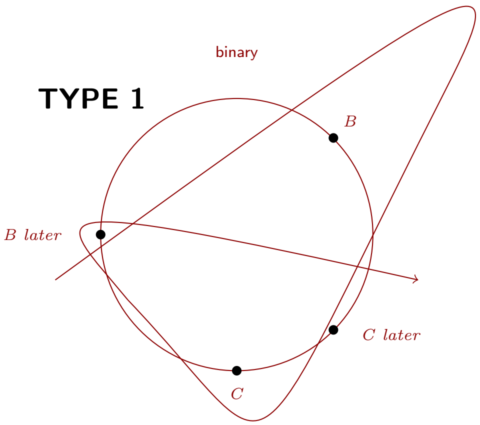

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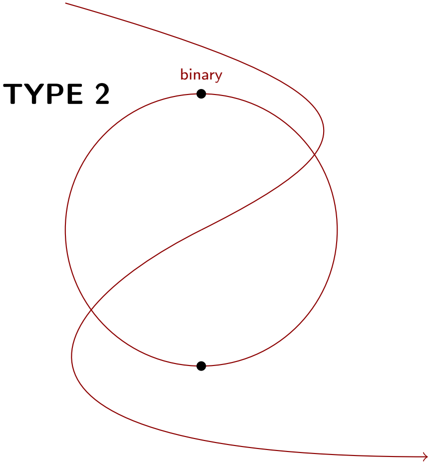

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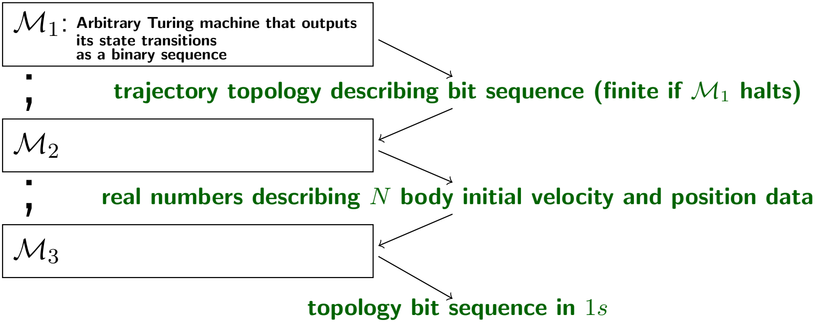

The halting revisited

Description of $\mathcal{M}_3$

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Philosophic question

If Warren's proof had been done in the beginning of the XX century, would the physicists have tried to reformulate Newtonian physics to make Physics simulatable and reestablish the Church-Turing thesis?

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Can we use a computational perspective (such like CT) as a refutation tool of a scientific theory? If not, what is the meaning of a non simulatable scientific theory?

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The Scientist Concept

The idea

A 'function' $\mathcal{M}$ embeds an algorithmic physical law whenever $\mathcal{M}$, on inputting the observations/measurements of an experiment, outputs a new 'programme' e, which simulates the instance of the physical law 'encoded' in the input (denoted by a text containing numbers).

The new 'programme' $\{e\}$, on input of some values assigned to the magnitudes of the involved physical concepts, outputs the predicted value of the derived physical concept for which the law was stated.

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$$pV = const$$

The scientist 'Boyle', on inputting text like this $\langle 5, \frac{2}{5}\rangle \# \langle 10, \frac{1}{5}\rangle \# \langle 20, \frac{1}{10}\rangle \# \ \ldots$, outputs the code e for the instance of Boyle's law with the constant 2.

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Scientists work with text!

Text et al.

1. A text T for a function is a map of type $\mathbb{N} \to [(\mathbb{N} \times \mathbb{N}) \cup \{\#\}]$, where the elements of the graph of a function ψ, $\langle t, \psi(t) \rangle$, for $t, \psi(t) \in \mathbb{N}$, are given separated by $\#$. By $T[t]$ we denote the sequence of the first t elements of T.
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Definition (Scientist)

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Depicting a scientist

Figure: For all $t \geq p$, scientist $\mathcal{M}$ on input $\psi[t]$ outputs code e of ψ.

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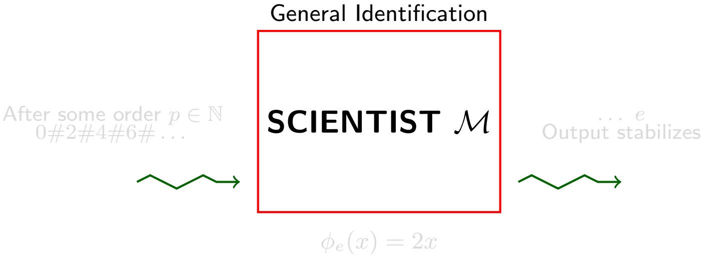

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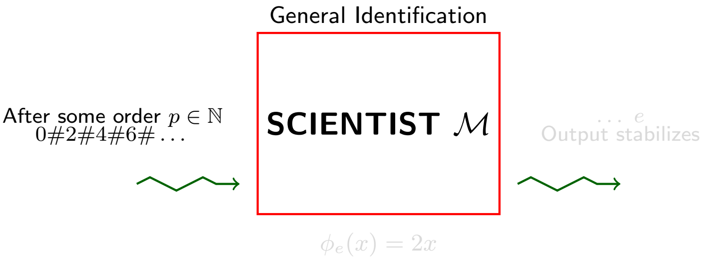

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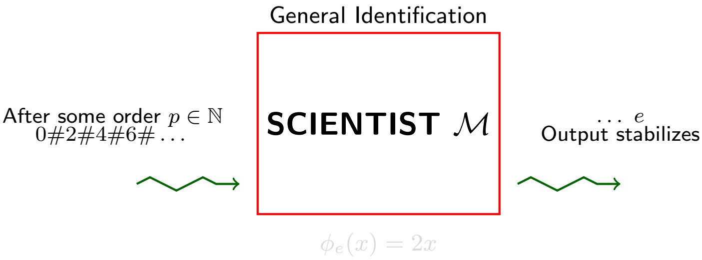

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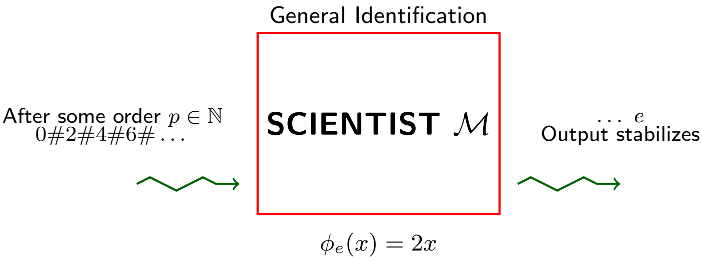

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Success for functions

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Let $\psi : \mathbb{N} \to \mathbb{N}$ a total function. We say that scientist $\mathcal{M}$ identifies ψ if there exists an $e \in \mathbb{N}$ and an order p such that, for $t \geq p$, $\mathcal{M}(\psi[t]) = e$ and $\phi_e = \psi$.

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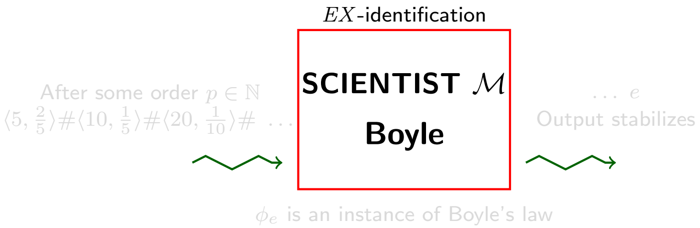

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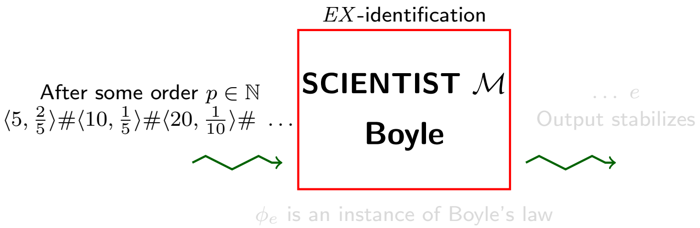

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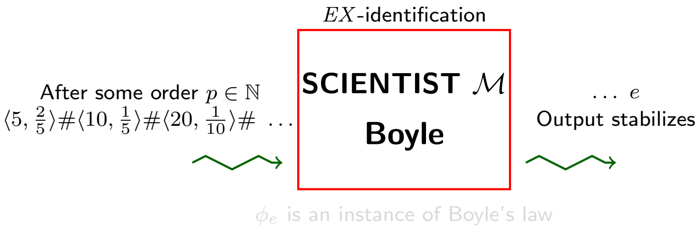

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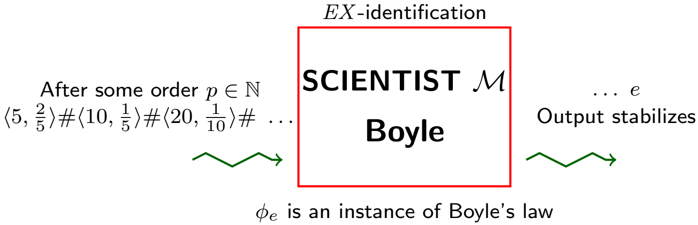

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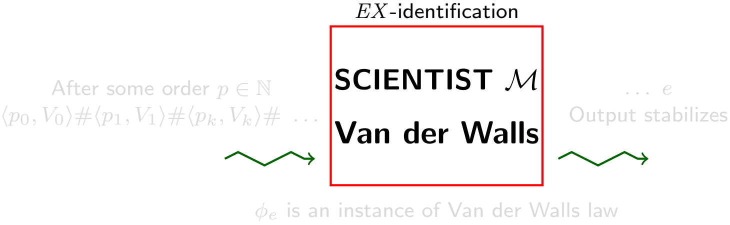

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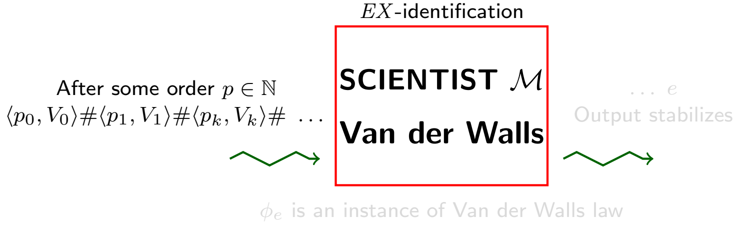

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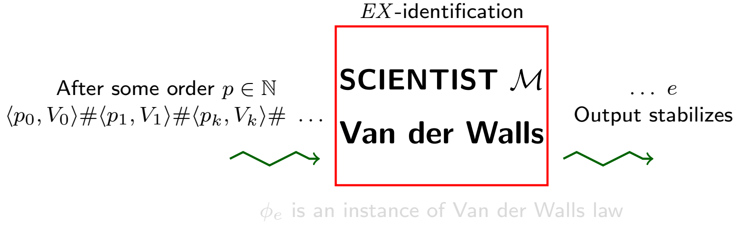

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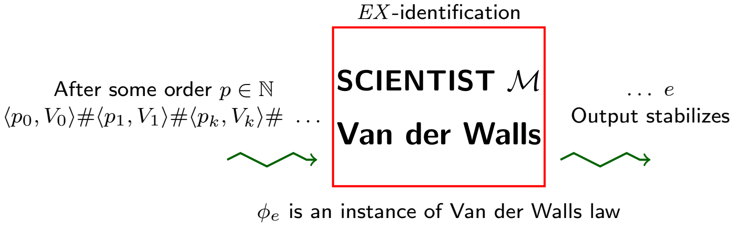

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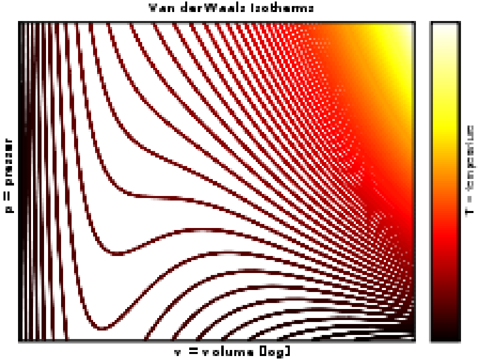

Figure: Van der Walls constitutive equation.

EXplain

Definition (EX^n-identification, Gold [Gol67], Case and Smith [CS78, CS83])

A set S of (total) recursive functions is said to belong to class EX^n, if there exists a scientist $\mathcal{M}$ such that, for each $\psi \in S$, there exists an order $p \in \mathbb{N}$ such that, for all $t \geq p$, $\mathcal{M}$ on input $\psi[t]$ outputs the same n-variant code for ψ.

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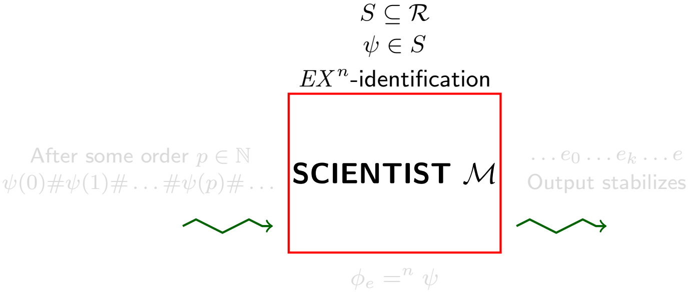

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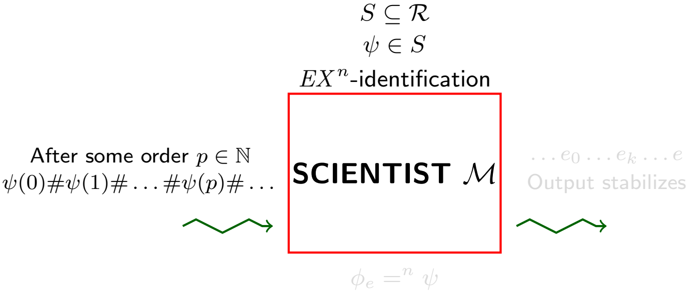

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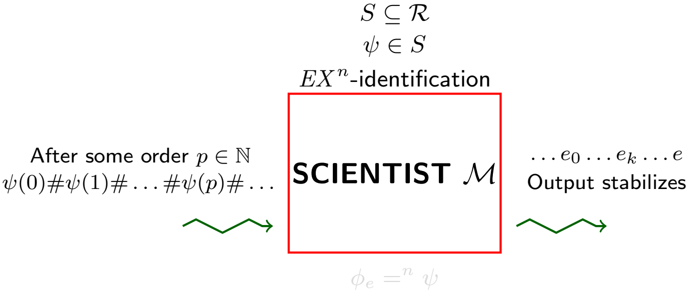

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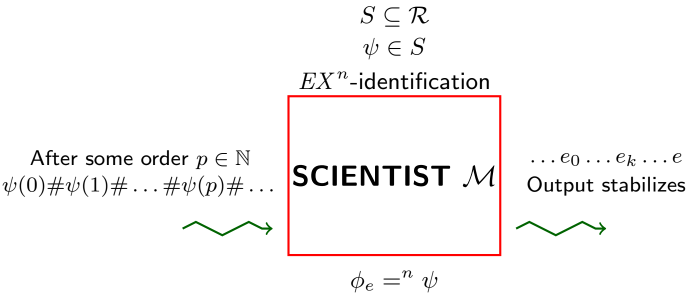

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Non-union theorem

Proposition (Blum and Blum [BB75], Jain et al. [JORS99])

The class EX is not closed under union.

Proof: We prove that $\mathcal{AEZ} \cup \mathcal{SD}$ is not EX-identifiable.

FUNCTION f:

Function $f(e, x : \mathbb{N}) : \mathbb{N}$;
Var σ : **list of** $\mathbb{N} \times \mathbb{N}$;
Begin
  $\sigma := \langle 0, e \rangle$;
  While $true$ **Do Begin**
    Find the least $\tau \in INIT$, $\tau \supset \sigma$, such that $\mathcal{M}(\tau) \neq \mathcal{M}(\sigma)$;
    $\sigma := \tau$;
    If $x \in dom(\widehat{\sigma})$ **Then Return** $\widehat{\sigma}(x)$
  End
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If $x = i$, **Then Output** $\psi(i)$, **Else Output** $\{\psi(0)\}(x)$.

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Non-collapsing hierarchy

Proposition (The hierarchy of scientists, mainly from Case and Smith [CS78, CS83], Harrington [CS83])

$$\mathcal{R} \notin EX = EX^0 \subset \cdots \subset EX^n \subset \cdots \subset EX^* \subset BC \not\ni \mathcal{R}$$

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Popper's Refutation Principle

Falsifiability

Popper states in the kernel of [Pop35]:

It has already been briefly indicated what rôle the basic statements play within the epistemological theory I advocate. We need them in order to decide whether a theory is to be called falsifiable, i.e. empirical [...] And we also need them for the corroboration of falsifying hypothesis, and thus for the falsification of theories [...]

Basic statements must therefore satisfy the following conditions: (a) From a universal statement without initial conditions, no basic statement can be deduced. On the other hand, (b) a universal statement and a basic statement can contradict each other. Condition (b) can only be satisfied if it is possible to derive the negation of a basic statement from the theory which it contradicts. From this and condition (a) it follows that a basic statement must have a logical form such that its negation cannot be a basic statement in its turn.

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Popper's refutability principle

Popper's refutability principle, Case and Smith [CS78, CS83]

The theory embedded in scientist $\mathcal{M}$ may not be refutable (!), for

1. It is not known if the instance ϕ_e is undefined on some y;
2. Programme $\{e\}$ on input y does not halt, i.e., one can not prepare any experimental apparatus to refute "theory $\mathcal{M}$ on y", given a basic statement such as $\phi_e(y) \neq \psi(y)$, where $\phi_e(y)$ is the prediction and $\psi(y)$ is the observation, since it is not even known with generality if $\{e\}(y)$ halts or not and, consequently, produce a prediction refutable by observation.

John Case adds in [Cas11]:

Hence, thanks to the unsolvability of the Halting Problem (see [Rog67]), Popper's Refutability Principle (in [Pop35]) is violated in a way Popper didn't consider ([CS78, CS83])!

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Bibliography I

[BB75] Lenore Blum and M. Blum. Toward a mathematical theory of inductive inference. *Information and Control*, 28:125–155, 1975.

[Cas11] John Case. Algorithmic scientific inference, within our computable expected reality, 2011. Technical Report, SUNY/Buffalo.

[CS78] John Case and Carl Smith. Anomaly hierarchies of mechanized inductive inference. In Richard J. Lipton, Walter A. Burkhard, Walter J. Savitch, Emily P. Friedman, and Alfred V. Aho, editors, *Proceedings of the 10th Annual ACM Symposium on Theory of Computing, May 1-3, 1978*, pages 314–319. ACM, San Diego, California, USA, 1978.

[CS83] John Case and Carl Smith. Comparison of identification criteria for machine inductive inference. *Theoretical Computer Science*, 25(2):193–220, 1983.

Bibliography II

[Ger91] Joseph Gerver. The existence of pseudo-collisions in the plane. *Journal of Differential Equations*, 89:1–68, 1991.

[Gol67] E. M. Gold. Language identification in the limit. *Information and Control*, 10:447–474, 1967.

[JORS99] Sanjay Jain, Daniel N. Osherson, James S. Royer, and Arun Sharma. *Systems That Learn. An Introduction to Learning Theory*. The MIT Press, second edition, 1999.

[Pop35] Karl R. Popper. *The Logic of Scientific Discovery*. Routledge, 1935. First English edition published in 1959 by Hutchinson & Co. First published by Routledge in 1992.

[Ric68] Daniel Richardson. Some undecidable problems involving elementary functions of a real variable. *The Journal of Symbolic Logic*, 33(4):514–520, 1968.

Bibliography III

[Rog67] Karl R. Rogers. *Theory of Recursive Functions and Effective Computability*. McGraw Hill, New York, 1967. Reprinted in 1987 by MIT Press.

[Smi06] Warren Smith. Church's thesis meets the N-body problem. *Applied Mathematics and Computation*, 178(1):154–183, 2006.

[Xia92] Zhihong Xia. The existence of noncollision singularities in Newtonian systems. *The Annals of Mathematics, Second Series*, 135(3):411–468, 1992.