# PDMA - Academic year 2012/13

# University of Minho and University of Aveiro
# Topics in Pure Mathematics - Module: Algebra and Geometry

### Lie Groups and Lie Algebras

### Homework 2

1. Show that the Lie algebra $su(2)$ is semisimple.

2. Let $\alpha, \beta$ be two roots of some semisimple complex Lie algebra.

   (a) If $\langle \alpha, \beta \rangle > 0$, show that then $\alpha - \beta$ is a root.

   (b) Conversely if $\langle \alpha, \beta \rangle < 0$, show that then $\alpha + \beta$ is a root.

   (c) Show that the $\alpha$ string containing $\beta$ has the form $\beta + n\alpha$ for $-n_- \leq n \leq n_+$, with $n_+, n_- \geq 0$. Show that there are no gaps. Furthermore show that $n_+ - n_- = -2\frac{\langle \beta,\alpha \rangle}{\langle \alpha,\alpha \rangle}$, and that the $\alpha$ string containing $\beta$ contains at most four roots.

3. Knowing its Dynkin diagram, determine the Cartan matrix for the Lie algebra $so(7)$. Using properties of root strings discussed in class (like those of exercise 2), determine all the roots of such algebra.

4. Recall that in the classification of simple Lie algebras we obtained four infinite series

   $$A_l = sl(l+1)\,(l \geq 1),\; B_l = so(2l+1)\,(l \geq 3),\; C_l = sp(2l)\,(l \geq 2),\; D_l = so(2l)\,(l \geq 4)$$

   and five exceptional cases $E_6, E_7, E_8, F_4, G_2$. Explain where to include in this list the simple Lie algebras $so(n), 3 \leq n \leq 6$ and $sp(2)$.

5. By analyzing their root systems, for each exceptional Lie algebra $E_6, E_7, E_8, F_4, G_2$ identify its largest possible (non-exceptional) subalgebra, justifying your answers.