PDMA - Academic year 2012/13
University of Minho and University of Aveiro
Topics in Pure Mathematics - Module: Algebra and Geometry

Lie Groups and Lie Algebras

Homework 1

The special unitary group in 3 dimensions SU(3) has $3^2 - 1 = 8$ generators, which can be represented by the Gell-Mann matrices $\lambda_i, i = 1, \ldots, 8$:

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

These generators obey a Lie algebra with structure constants f_{ijk} (up to a factor $2i$):

$$[\lambda_i, \lambda_j]_- = 2i \sum_{k=1}^{8} f_{ijk} \lambda_k.$$

The anticommutator, defined as $[\lambda_i, \lambda_j]_+ := \lambda_i \lambda_j + \lambda_j \lambda_i$, is used to define the coefficients d_{ijk} :

$$[\lambda_i, \lambda_j]_+ =: \frac{4}{3} \delta_{ij} \mathbf{1} + 2 \sum_{k=1}^{8} d_{ijk} \lambda_k,$$

$\mathbf{1}$ being the 3×3 identity matrix.

SU(3) has two Casimir operators, which can be defined as

$$C_1 := \frac{1}{4} \sum_{i=1}^{8} \lambda_i^2, \; C_2 := \frac{1}{8} \sum_{i=1}^{8} \sum_{j=1}^{8} \sum_{k=1}^{8} d_{ijk} \lambda_i \lambda_j \lambda_k.$$

1. Show that the eight generators λ_i are linearly independent and span the space of all hermitean traceless 3×3 matrices.

2. Show that
$$\mathrm{Tr}\, \lambda_i \lambda_j = 2\delta_{ij}.$$
(Don't forget that $\mathrm{Tr}\, \mathbf{1} = 3$ and $\mathrm{Tr}\, \lambda_i = 0$.)

3. Prove that $d_{ijk} = \frac{1}{4} \mathrm{Tr} \left([\lambda_i, \lambda_j]_+ \lambda_k \right)$. Using this relation, show that the coefficients d_{ijk} are totally symmetric.

4. Prove that $f_{ijk} = \frac{1}{4i} \mathrm{Tr} \left([\lambda_i, \lambda_j]_- \lambda_k \right)$. Using this relation, show that the structure constants f_{ijk} are totally antisymmetric.

5. Compute the structure constants f_{156}, f_{148} and the coefficients d_{118}, d_{778}, using the given explicit representation of the matrices λ_i.

6. Derive the identities

$$\sum_{m=1}^{8} (f_{plm} f_{mkq} + f_{klm} f_{mqp} + f_{pkm} f_{mql}) = 0;$$

$$\sum_{m=1}^{8} (f_{pkm} d_{mlq} + f_{qkm} f_{mlp} + f_{lkm} d_{mpq}) = 0.$$

7. Verify that C_1, C_2 are indeed Casimir operators of SU(3).

8. Prove the following relations:

$$C_1 = -\frac{i}{12} \sum_{i=1}^{8} \sum_{j=1}^{8} \sum_{k=1}^{8} f_{ijk} \lambda_i \lambda_j \lambda_k;$$

$$C_2 = C_1 \left(2C_1 - \frac{11}{6} \right).$$

For each answer, you may always use results from the previous questions.