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Aspects of Gravity's Rainbow in Black Hole Entropy

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Black Hole Entropy & Gravity's Rainbow

[J. D. Bekenstein, Phys. Rev. D 7, 949 (1973). S. W. Hawking, Comm. Math. Phys. 43, 199 (1975).]

Bekenstein-Hawking entropy $$S_{BH} = \frac{A}{4l_P^2} \qquad T_H = \frac{\hbar \kappa_0}{2\pi}$$

$\kappa_0 \quad \Leftrightarrow \quad$ surface gravity

The entropy is simply $$S=\beta^2 \frac{\partial F}{\partial \beta} \Rightarrow S=\frac{16\pi^3}{90\beta^3} \int_{r_0+h}^{R} \frac{\exp\left(2\Lambda\left(r\right)\right)}{\left(1-\frac{b\left(r\right)}{r}\right)^2} r^2 dr$$

$$S=\frac{16\pi^3}{90\beta^3}\frac{r_0^2}{4\kappa_0^3}\frac{\exp\left(-\Lambda\left(r_0\right)\right)}{\alpha^2}=\frac{A}{90}\left(\frac{T}{\kappa_0/2\pi}\right)^3\frac{\exp\left(-\Lambda\left(r_0\right)\right)}{4\pi\alpha^2}$$

$$Identification \quad \frac{\exp\left(-\Lambda\left(r_0\right)\right)}{90\pi\alpha^2}=\frac{1}{l_P^2} \Rightarrow S=\frac{A}{4l_P^2}$$

Brick Wall

't Hooft NPB 256, 727 (1985)

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Eliminating the brick wall from QFT Procedures

- Renormalization of Newton's constant
- [L. Susskind and J. Uglum, Phys. Rev. D50, 2700 (1994). J. L. F. Barbon and R. Emparan, Phys. Rev. D52, 4527 (1995) 1995), hep-th/9502155. E. Winstanley, Phys. Rev. D63, 084013 (2001) 2001), hep-th/0011176.]
- Pauli-Villars regularization
- [J.-G. Demers, R. Lafrance and R. C. Myers, Phys. Rev. D52, 2245 (1995), gr-qc/9503003. D. V. Fursaev and S. N. Solodukhin, Phys. Lett. B365, 51 (1996), hep-th/9412020. S. P. Kim, S. K. Kim, K.-S. Soh and J. H. Yee, Int. J. Mod. Phys. A12, 5223 (1997) gr-qc/9607019.]

Eliminating the brick wall using Generalized Uncertainty Principle

- X. Li, Phys. Lett. B 540, 9 (2002), gr-qc/0204029.
- Z. Ren, W. Yue-Qin and Z. Li-Chun, Class. Quant. Grav. 20 (2003), 4885.
- G. Amelino-Camelia, Class.Quant.Grav. 23, 2585 (2006), gr-qc/0506110.
- G. Amelino-Camelia, Gen.Rel.Grav. 33, 2101 (2001), gr-qc/0106080.

$$\Delta x \Delta p \geq \hbar \frac{\lambda_P^2}{\hbar}(\Delta p)^2 \qquad \lambda_P \Leftrightarrow \text{Planck Length}$$

$$\hbar \Leftrightarrow \text{Planck Constant}$$

$$\frac{d^3x d^3p}{(2\pi\hbar)^3 \left(1+\lambda_P^2 p^2\right)^3}$$

Gravity’s Rainbow

Doubly Special Relativity

G. Amelino-Camelia, Int.J.Mod.Phys. D 11, 35 (2002); gr-qc/001205.
G. Amelino-Camelia, Phys.Lett. B 510, 255 (2001); hep-th/0012238.

$$E^2 g_1^2\left(E/E_P\right) - p^2 g_2^2\left(E/E_P\right) = m^2$$

$$\lim_{E/E_P\to 0} g_1\left(E/E_P\right) = \lim_{E/E_P\to 0} g_2\left(E/E_P\right) = 1$$

Curved Space Proposal → *Gravity’s Rainbow*
[J. Magueijo and L. Smolin, Class. Quant. Grav. 21, 1725 (2004) arXiv:gr-qc/0305055].

$$G_{\mu\nu} = 8\pi G\left(E\right) T_{\mu\nu}\left(E\right) + g_{\mu\nu}\Lambda\left(E\right)$$

$$G\left(E\right) \to G\left(0\right) \quad \text{when} \quad E \ll E_P$$

$$\Lambda\left(E\right) \to G\left(0\right) \quad \text{when} \quad E \ll E_P$$

Gravity's Rainbow

Curved Space Proposal → *Gravity's Rainbow*
[J. Magueijo and L. Smolin, Class. Quant. Grav. 21, 1725 (2004) arXiv:gr-qc/0305055].

$$ds^2 = -\left(1-\frac{2MG(0)}{r}\right)\frac{d\tilde{t}^2}{g_1^2(E/E_P)} + \frac{d\tilde{r}^2}{\left(1-\frac{2MG(0)}{r}\right)g_2^2(E/E_P)} + \frac{\tilde{r}^2}{g_2^2(E/E_P)}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Generalization

$$ds^2 = -\exp(-2\Lambda(r))\frac{dt^2}{g_1^2(E/E_P)} + \frac{dr^2}{\left(1-\frac{b(r)}{r}\right)g_2^2(E/E_P)} + \frac{r^2}{g_2^2(E/E_P)}d\theta^2 + \frac{r^2}{g_2^2(E/E_P)}\sin^2\theta d\varphi^2$$

- $b(r)$ is the shape function
- $\Lambda(r)$ is the redshift function

$$b(r_0) = r_0 \quad r \in [r_0, +\infty)$$

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[R.Garattini P.L.B. **B685 (2010) 329** e-Print: **arXiv:0902.3927 [gr-qc]]**

From the equation of motion, we can define an r-dependent radial wave number

$$k^2(r,l,E)=\frac{1}{1-\frac{b(r)}{r}}\left[\exp(2\Lambda(r))\frac{E^2\tilde{h}^2(E/E_P)}{1-\frac{b(r)}{r}}-\frac{l(l+1)}{r^2}\right]$$

$$\tilde{h}(E/E_P)=\frac{g_1(E/E_P)}{g_2(E/E_P)}$$

Free energy $\quad F=\frac{2}{\pi\beta}\int_0^\infty \ln\left(1-\exp(-\beta E)\right)\frac{d}{dE}\left(\frac{1}{3}E^3\tilde{h}^3(E/E_P)\right)dE\int_{r_0+h}^{R}\frac{\exp(3\Lambda(r))}{\left(1-\frac{b(r)}{r}\right)^2}r^2dr$

Assumption $\Rightarrow r_0+h=r_0+h(E/E_P)=r_0\left(1+\sigma(E/E_P)\right)$

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In proximity of the throat, we consider the approximate free energy

Free energy $$F_{r_0} = \frac{2r_0^4}{\pi\beta}\frac{\exp\left(3\Lambda\left(r_0\right)\right)}{\left(1-b'\left(r_0\right)\right)^2}\int_0^\infty \frac{\ln\left(1-\exp\left(-\beta E\right)\right)}{r_0\sigma\left(E/E_P\right)}\frac{d}{dE}\left(\frac{1}{3}E^3\tilde{h}^3\left(E/E_P\right)\right)dE$$

An integration by parts is possible if $\tilde{h}\left(E/E_P\right)$ is rapidly convergent

$\rightarrow$ Good choice $\tilde{h}\left(E/E_P\right) = \exp\left(-E/E_P\right)$

Assumption on $\sigma\left(E/E_P\right) = \tilde{h}^\delta\left(E/E_P\right)\left(\frac{E}{E_P}\right)^\alpha$

Property $\Rightarrow \sigma\left(E/E_P\right) \rightarrow 0$ when $E/E_P \rightarrow 0$

Two interesting cases

Case a) $\delta=0$ $\alpha>0$

Case b) $\delta>0$ $\alpha>0$

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$$S_{BH}=\frac{A}{4l_P^2}$$

Case a)

In the limit $\epsilon E_P >> 1$

$$S=\frac{A_{r_0}E_P^2}{4}\frac{\exp\left(2\Lambda\left(r_0\right)\right)}{1-b'\left(r_0\right)}\frac{2}{9\pi}$$

To recover the area law, we have to set $\dfrac{\exp\left(2\Lambda\left(r_0\right)\right)}{1-b'\left(r_0\right)}=\dfrac{9\pi}{2}$

This corresponds to a changing of the time variable with respect to the Schwarzschild time.

The internal energy $U=\dfrac{M}{4}$ where we have used

$r_0=2MG$ and $\beta=8\pi MG$

Discrepancy of a factor of 3/2 with the 't Hooft result

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Work in progress….also rotations can be included

Rotating
Heat bath

$$ds^2(E) = -\frac{dt^2}{g_1^2(E)} + \frac{dr^2}{g_2^2(E)} + \frac{r^2 d\phi^2}{g_2^2(E)} + \frac{dz^2}{g_2^2(E)}.$$

Comoving Frame

$$ds^2(E) = -\left(1-\Omega_0^2 r^2\right)\frac{dt^2}{g_1^2(E)} - \frac{2\Omega_0 r^2 d\phi' dt}{g_1(E)\, g_2(E)} + \frac{dr^2}{g_2^2(E)} + \frac{r^2 d\phi'^2}{g_2^2(E)} + \frac{dz^2}{g_2^2(E)}.$$

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Kerr metric

$$ds^2 = g_{tt}\frac{dt^2}{g_1^2(E/E_P)} + 2g_{t\phi}\frac{dtd\phi}{g_1(E/E_P)g_2(E/E_P)} + g_{\phi\phi}\frac{d\phi^2}{g_2^2(E/E_P)} + g_{rr}\frac{dr^2}{g_2^2(E/E_P)} + g_{\theta\theta}\frac{d\theta^2}{g_2^2(E/E_P)}$$

$$g_{tt} = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}, \qquad g_{t\phi} = -\frac{a\sin^2\theta\,(r^2+a^2-\Delta)}{\Sigma},$$

$$g_{\phi\phi} = \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta, \qquad g_{rr} = \frac{\Sigma}{\Delta}, \qquad g_{\theta\theta} = \Sigma,$$

$$\Delta = r^2 - 2MGr + a^2, \qquad \Sigma = r^2 + a^2\cos^2\theta$$

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Kerr metric → ZAMO

$$ds^2 = -N^2 \frac{dt^2}{g_1^2\left(E/E_P\right)} + g_{\phi\phi}\frac{d\tilde{\phi}^2}{g_2^2\left(E/E_P\right)} + g_{rr}\frac{dr^2}{g_2^2\left(E/E_P\right)} + g_{\theta\theta}\frac{d\theta^2}{g_2^2\left(E/E_P\right)}$$

$$N^2 = -\frac{\Delta\sin^2\theta}{g_{\tilde{\phi}\tilde{\phi}}} \to 0 \text{ when } r \to r_+$$

Free energy

$$\mathrm{F} = \frac{1}{8\pi^2\beta}\int d\theta d\tilde{\phi}\int_0^\infty \ln\left(1-\exp\left(-\beta E\right)\right)\frac{d}{dE}\left(\frac{1}{3}E^3\tilde{h}^3\left(E/E_P\right)\right)dE\int_{r_++h}^{R}\left(-g^{tt}\right)^{\frac{3}{2}}\sqrt{g_{rr}g_{\theta\theta}g_{\tilde{\phi}\tilde{\phi}}}\,dr$$

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Kerr metric → ZAMO

Close to r_+

$$\int_{r_+\left(1+\sigma\left(E/E_P\right)\right)}^{r_1} \left(-g^{tt}\right)^{\frac{3}{2}} \sqrt{g_{rr}\, g_{\theta\theta}\, g_{\tilde{\phi}\tilde{\phi}}}\, dr \simeq \frac{\left(r_+^2 + a^2\right)^4 \sin\theta}{r_+ \sigma\left(E/E_P\right)\left(r_+ - r_-\right)^2 \Sigma_+}$$

Property $\Rightarrow \sigma\left(E/E_P\right) \to 0$ when $E/E_P \to 0$ Assumption on $\sigma\left(E/E_P\right) = \tilde{h}^{\delta}\left(E/E_P\right)\left(\frac{E}{E_P}\right)^{\alpha}$

$For\ \delta=0\ \ \alpha = 3$

Free energy close to r_+

$$\mathrm{F} \simeq -\frac{1}{8\pi^2\beta}\int d\theta\, d\tilde{\phi}\left[\varsigma\left(2,1+\frac{3}{\beta E_P}\right) + \frac{\beta E_P}{3}\left(\gamma + \Psi\left(1+\frac{3}{\beta E_P}\right)\right)\right]\frac{\left(r_+^2 + a^2\right)^4 \sin\theta}{r_+\left(r_+ - r_-\right)^2 \Sigma_+}$$

Conclusions

- Application of Gravity's Rainbow can be considered to compute divergent quantum observables.
- Neither Standard Regularization nor Renormalization are required. This also happens in NonCommutative geometries and G.U.P. modifications
- The thermodynamical observables can be computed in the context of spherically symmetric backgrounds.
- Application to rotating black holes….In Progress….