

# A stellar model with diffusion in general relativity (arXiv:1602.02663 [gr-qc])

Artur Alho  
(Joint work with Simone Calogero)

Center for Mathematical Analysis, Geometry and Dynamical Systems, Technical University of Lisbon

IX black-holes workshop, Guimarães - December 19th, 2016



## Diffusion in General Relativity

- Kinetic theory based on a Fokker-Planck equation for the particle density in the phase-space:  
S. Calogero, JCAP 11/2011, 016 (2011)
- Fluid theory which is the formal macroscopic limit of the kinetic theory: S. Calogero J. Geom. Phys. **62**, 2208–2213 (2012)

- The energy-momentum tensor and energy current for a perfect fluid

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)$$

$$J^\mu = n u^\mu$$

- For a perfect-fluid undergoing velocity diffusion

$$\nabla_\mu T^{\mu\nu} = \sigma J^\nu$$

$$\nabla_\mu J^\mu = 0$$

- $\sigma$  is the diffusion constant and measures the average energy transferred per unit time from the background substance to a fluid particle.
- Projecting parallel and orthogonal to  $u^\mu$

$$\nabla_\mu(\rho u^\mu) + p \nabla_\mu u^\mu = \sigma n,$$

$$(\rho + p) u^\mu \nabla_\mu u^\nu + u^\nu u^\mu \nabla_\mu p + \nabla^\nu p = 0,$$

$$\nabla_\mu(n u^\mu) = 0.$$

- First-law of thermodynamics:

$$u^\mu \nabla_\mu s = \frac{\sigma}{T},$$

- In presence of diffusion  $T_{\mu\nu}$  is not divergence-free. Incompatibility with the twice contracted Bianchi identities  $\nabla_\mu G^{\mu\nu} = 0$ .
- Add a matter field which interacts with the fluid particles restoring the local conservation of energy
- The new matter field plays the role of a background medium in which particles undergo diffusion
- The simplest model for this medium is a vacuum-energy described by a cosmological scalar field (varying  $\Lambda$ )

$$G_{\mu\nu} + \phi g_{\mu\nu} = T_{\mu\nu}$$

- The diffusion equation is

$$\nabla_\mu \phi = \sigma J_\mu$$

- When  $\sigma = 0$  the model reduces to the Einstein-Euler system with cosmological constant  $\Lambda$ .
- A. Alho, S. Caloger, M. P. Ramos and A. J. Soares: Dynamics of Robertson-Walker spacetimes with diffusion, *Annals of Physics* **204** (2015)
- Stellar models in Spherical Symmetry

## The Interior

- Comoving system of coordinates:

$$g = -e^{2\Phi(t,R)} dt^2 + e^{2\Psi(t,R)} dR^2 + r^2(t,R) d\Omega^2,$$

$$u = -e^{-\Phi(t,R)} \partial_t,$$

where  $d\Omega^2 = d\theta^2 + \sin^2\theta d\psi^2$  is the standard metric on  $S^2$ .

- Fix  $\Phi(t, R_b) = 0$ , so that  $t$  is the proper time of observers at rest with respect to the boundary of the star
- Fix  $r(0, R) = R$ , so that the comoving radius  $R$  coincides initially, i.e., at time  $t = 0$ , with the radius function of the group orbits.

### Theorem

Let  $p = 0$  and let  $(g, \rho, n, u, \phi)$  be a spherically symmetric solution. Then  $\rho, n, \phi$  are functions of  $t \in [0, T)$  only and there exist a positive function  $a : [0, T) \rightarrow (0, \infty)$  and a constant  $k \in \mathbb{R}$  such that

$$g = -dt^2 + a(t)^2 \left( \frac{dR^2}{1 - kR^2} + R^2 d\Omega^2 \right),$$

$$u = -\partial_t.$$

- Robertson-Walker geometry! By rescaling the radial coordinate we may assume that  $k \in \{-1, 0, 1\}$ ,  $R < R_b$ , where  $R_b < 1$  for  $k = 1$ .

- From now on we consider a pressureless fluid (dust).
- The conservation of the particle number density gives  $n(t) = \frac{a(0)^3 n(0)}{a(t)^3}$ , while  $a(t), \rho(t), \phi(t)$  satisfy

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{3}(\rho + \phi) \quad , \quad \frac{\ddot{a}}{a} = -\frac{1}{6}\rho + \frac{1}{3}\phi,$$

$$\dot{\phi} = -3\frac{\beta}{a^3}, \quad , \quad \dot{\rho} = -3\rho\frac{\dot{a}}{a} - \dot{\phi},$$

where  $\beta = \sigma n(0)a(0)^3/3$ .

- Two classes of solutions: “Expanding” and “Collapsing”
- Collapsing (dust) solutions behave like the Friedmann-Lemaître diffusion-free solution in the limit toward the singularity.
- This implies that the spacelike singularity at  $t = t_s$  is a curvature singularity,
- The spacetime is inextendible beyond the spacelike hypersurface  $t = t_s$  and no outgoing light ray can emanate from the singularity.

## Corollary

*There exist no naked singularities in the spherical collapse of dust clouds undergoing diffusion in a cosmological scalar field.*

- Let  $\chi = g^{ab}(\partial_a r)(\partial_b r)$ ,  $a = 0, 1$ :

$$\chi(t, R) = 1 - kR^2 - \dot{a}(t)^2 R^2 = (\sqrt{1 - kR^2} + \dot{a}(t)R)(\sqrt{1 - kR^2} - \dot{a}(t)R).$$

- Trapped region:  $\chi < 0$ ,
- Regular region:  $\chi > 0$ .
- Apparent horizon:  $\chi = 0$
- Introduce local mass function

$$m(t, R) = \frac{\rho(t)}{6} r^3, \quad r(t, R) = a(t)R$$

- In terms of  $m$  and  $\phi$

$$\chi = 1 - \frac{2m}{r} - \frac{\phi}{3} r^2, \quad r(t, R) = a(t)R.$$

- The mass function obeys

$$\partial_t m = \frac{R^3}{2} \beta > 0.$$

- As the mass of the interior is not conserved, the exterior of the star cannot be static.

## The Exterior

- Bondi coordinates:

$$g_{\text{ext}} = -A(w, r)B(w, r)dw^2 + 2\varepsilon A(w, r)dwdr + r^2 d\Omega^2,$$

where  $\varepsilon = \pm 1$ .

- $\varepsilon = 1$ ,  $w$  is the ingoing (advanced) null coordinate.
- $\varepsilon = -1$ ,  $w$  is the outgoing (retarded) null coordinate.
- Boundary of the star  $\Sigma$

$$\Sigma : r = r_{\Sigma}(w),$$

so that  $r > r_{\Sigma}(w)$ .

- $\Sigma$  is assumed to be timelike:

$$A(w, r(w))(B(w, r(w)) - 2\varepsilon \dot{r}_{\Sigma}(w)) > 0, \quad \dot{r}_{\Sigma} = \frac{dr_{\Sigma}}{dw},$$

i.e., the first fundamental form of  $\Sigma$  has the signature  $(-, +, +)$ .

- Time orientation of spacetime does not change across the boundary

$$dt/dw > 0$$

- The two metrics  $g_{\text{int}}$  and  $g_{\text{ext}}$  may be matched on  $\Sigma$  if and only if they satisfy the junction conditions that they induce the same first and second fundamental form on  $\Sigma$ ,

## Theorem

The metrics satisfy the junction conditions if and only if

(a) The transformation of variable  $t = t(w)$  satisfies

$$\dot{t}(w) = \left[ A(w, r_\Sigma(w)) \left( B(w, r_\Sigma(w)) - 2\varepsilon \dot{r}_\Sigma(w) \right) \right]^{1/2};$$

(b) There holds

$$r_\Sigma(w) = a(t(w))R_b, \quad \dot{r}_\Sigma(w) = \dot{a}(t(w))R_b(\sqrt{1 - kR_b^2} - \varepsilon \dot{a}(t(w))R_b);$$

(c) There holds

$$B(w, r_\Sigma(w)) = A(w, r_\Sigma(w))(1 - kR_b^2 - \dot{a}(t(w))^2 R_b^2);$$

(d) There holds  $Q(w) = 0$ , where

$$Q(w) = \left[ (B - 2\varepsilon \dot{r}_\Sigma(w))((B - \varepsilon \dot{r}_\Sigma(w))\partial_r A + \varepsilon \partial_w A) \right. \\ \left. - A(2\ddot{r}_\Sigma(w) + (B - 3\varepsilon \dot{r}_\Sigma(w))\partial_r B - \varepsilon \partial_w B) \right]_{r=r_\Sigma(w)}.$$



- Assuming  $A(w, r) = 1$ :

## Corollary

The exterior metric can be chosen to be the Vaidya type metric with variable cosmological constant given by

$$g_{\text{ext}} = -\left(1 - \frac{2M(w)}{r} - \frac{\Lambda(w)}{3}r^2\right)dw^2 + 2\epsilon dw dr + r^2 d\Omega^2,$$

where  $M(w), \Lambda(w)$  are given by

$$M(w) = m(t(w), R_b) \quad \text{and} \quad \Lambda(w) = \phi(t(w)).$$

- The generalized Vaidya metric solves the Einstein equation with cosmological scalar field  $\phi = \Lambda(w)$  and energy-momentum tensor

$$T = \tilde{\rho} dw^2,$$

where

$$\tilde{\rho} = \frac{\epsilon}{r^2} \left( 2 \frac{dM}{dw} + \frac{r^3}{3} \frac{d\Lambda}{dw} \right) = \frac{\epsilon \beta \frac{dt}{dw}}{r^2 a(t(w))^3} [r_{\Sigma}^3(w) - r^3], \quad r \geq r_{\Sigma}(w)$$

## Proposition

$\tilde{\rho}(w, r_{\Sigma}(w)) = 0$ . Moreover the weak energy condition  $\tilde{\rho}(w, r) > 0$ , for all  $r > r_{\Sigma}(w)$ , holds only for the outgoing Vaidya metric  $\epsilon = -1$ .

## Self-Similar solution

- explicit solution:

$$a(t) = \delta_k t \quad , \quad \phi(t) = \frac{3\beta}{2\delta_k} a(t)^{-2} \quad , \quad \rho(t) = \frac{3\beta}{\delta_k} a(t)^{-2},$$

where  $\delta_k$  is the real solution of the polynomial equation

$$\delta^3 + k\delta - \frac{3\beta\gamma}{2} = 0.$$

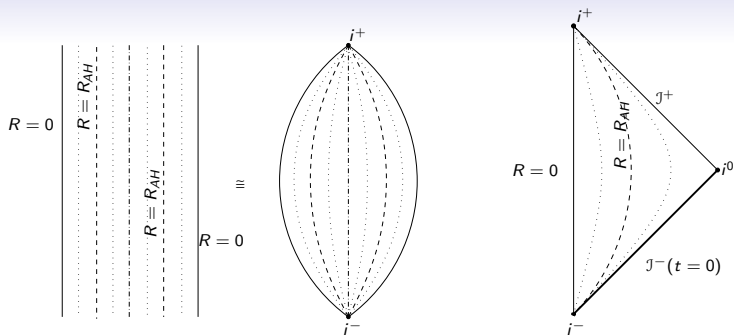
- Note that  $\delta_k > 0$ , for all  $k = 0, \pm 1$ .
- Towards the past: curvature singularity
- Toward the future: the solution is forever expanding ( $\dot{a} > 0$ ) without acceleration ( $\ddot{a} = 0$ ).
- Apparent horizon:

$$R_{\text{AH}} = \frac{1}{\sqrt{\delta_k^2 + k}}.$$

- The local mass of the interior is

$$m(t, R) = \frac{\beta}{2} R^3 t,$$

hence  $m(t, R) \rightarrow +\infty$  as  $t \rightarrow +\infty$ .



(a) Conformal diagram and bounded conformal diagram for  $k = 1$ . The solid lines correspond to the boundary  $R = 0$  and the dashdotted line to the equator  $R = 1$ . The remaining dotted lines are curves of constant  $R$ , for  $0 < R < 1$ , while the dashed lines represent the apparent horizon  $R = R_{AH}$ . In this case a suitable matching surface is given by the curve  $R = R_b < 1$ .

(b) Bounded conformal diagram  $k = 0$  and  $k = -1$ . The dotted lines are curves of constant  $R$  and the dashed line is the apparent horizon  $R = R_{AH}$ . The thick solid line corresponds to a Big-Bang type (null) singularity.

**Figure:** Penrose diagrams for the expanding (at constant rate) interior solution. Each point represents a 2-sphere of radius  $r = a(t)R$ . As usual,  $i^-$  and  $i^+$  represent past and future timelike infinity respectively, and  $i^0$  corresponds to spacelike infinity. Also,  $J^-$ ,  $J^+$  denote past and future null infinity respectively.

- Let  $R = R_b$  the boundary of the star. We distinguish three cases:

- (i)  $R_b > R_{\text{AH}}$ ; the interior has an apparent horizon in this case.
- (ii)  $R_b = R_{\text{AH}}$ ; the boundary of the star coincides with the apparent horizon.
- (iii)  $R_b < R_{\text{AH}}$ ; the interior has no apparent horizon.

- The matching conditions give

$$u = C_k t, \quad r_{\Sigma}(u) = x_{\Sigma} u, \quad x_{\Sigma} = \frac{R_b \delta_k}{C_k}, \quad C_k = \frac{1}{\sqrt{1 - kR_b^2 + \delta_k R_b}} > 0,$$

- the exterior metric becomes

$$g_{\text{ext}} = -\left(1 - \lambda_1 \frac{u}{r} - \lambda_2 \frac{r^2}{u^2}\right) du^2 - 2du dr + r^2 d\Omega^2, \quad r > x_{\Sigma} u,$$

where

$$\lambda_1 = \frac{\beta R_b^3}{C_k}, \quad \lambda_2 = \frac{\beta C_k^2}{2\delta_k^3}.$$

- Curvature singularity at  $u = 0$ , for its Kretschmann scalar  $K = \text{Riem}^2$  is given by

$$K = \frac{12\lambda_1^2 u^2}{r^6} + \frac{24\lambda_2^2}{u^4}.$$

- The apparent horizons: hypersurfaces where  $B(u, r) = 0$ ,

$$B(u, r) = h\left(\frac{r}{u}\right), \quad h(x) = 1 - \frac{\lambda_1}{x} - \lambda_2 x^2.$$

## • Proposition

The following holds:

- (1) In case (i) there is no apparent horizon in the exterior region and  $B(u, r) < 0$  for all  $r > r_{\Sigma}(u)$ ,  $u > 0$ .
- (2) In case (ii) the apparent horizon in the exterior coincides with the apparent horizon of the interior, as well as with the matching surface:

$$r_{\text{AH}}(u) = r_{\Sigma}(u) \equiv R_{\text{AH}} = R_b.$$

Moreover  $B(u, r) < 0$  for all  $r > r_{\Sigma}(u)$ .

- (3) In case (iii) there exists  $x_{\text{AH}} > x_{\Sigma}$  such that the metric in the exterior has an apparent horizon at  $r = x_{\text{AH}}u$ . Moreover  $B(u, r) > 0$  for  $r_{\Sigma}(u) < r < x_{\text{AH}}u$  and  $B(u, r) < 0$  for  $r > x_{\text{AH}}u$ .

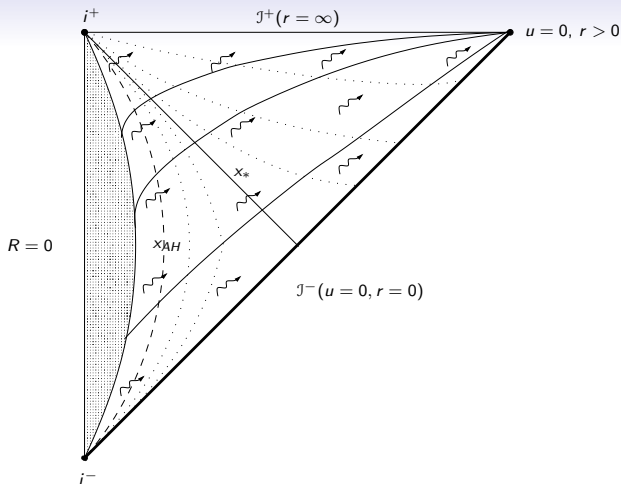
## Proof.

For the proof it suffices to notice that the function  $h(x)$  attains its maximum at  $x = x_{\Sigma}$  and  $h(x_{\Sigma}) = 1 - (R_b/R_{\text{AH}})^2$ . □

- The model under discussion is self-similar, with the lines  $r = xu$ ,  $x > x_{\Sigma}$ , on the  $(t, r)$ -plane being tangent to the homothetic vector field in the exterior. We call such curves homothetic curves.

## Theorem

There exists  $x_* > x_{\Sigma}$  such that the homothetic curve  $r = xu$  is spacelike for  $x > x_*$ , null for  $x = x_*$  and timelike for  $x_{\Sigma} < x < x_*$ . In case (iii) there holds  $x_* > x_{\text{AH}}$ . Moreover the homothetic curve  $r = x_*u$  is the first ingoing radial null geodesics that escapes to null-infinity.



**Figure:** Penrose diagram for the self-similar stellar solution with diffusion in case (iii) showing the (homothetic) apparent horizon  $x_{AH}$  in the exterior region (dashed timelike curve). The null homothetic (cosmological) horizon  $x_*$  is denoted by a solid line, while the other timelike and spacelike homothetic curves are denoted by dotted lines. The remaining solid lines are constant  $r$  curves which are spacelike, becoming null at the apparent horizon, and are timelike afterwards. The thick solid line corresponds to the singularity. As usual,  $i^-$  and  $i^+$  represent past and future timelike infinity respectively, and  $\mathcal{J}^-$ ,  $\mathcal{J}^+$  denote past and future null infinity respectively.