New conserved currents for vacuum space-times in dimension four with a Killing vector

Alfonso García-Parrado Gómez-Lobo*

*Dep. Física Teórica, Universidad del País Vasco, Bilbao (Spain)

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Bibliography:

A. García-Parrado, *New conserved currents for vacuum space-times in dimension four with a Killing vector*,
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Outline

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The role of conserved currents

- The energy-momentum conservation principle can be mathematically expressed by means of a conserved current and its conserved charge.
- A conserved current is a vector field $\vec{\boldsymbol{Z}}$ such that $\nabla_\mu Z^\mu = 0$. Use Gauß Theorem to construct conserved charges.

$$0 = \int_\Omega \nabla_\mu Z^\mu \boldsymbol{\eta} = \int_{\mathcal{H}} (n_\mu Z^\mu) d\mathcal{H} \; , \quad \mathcal{H} \equiv \partial\Omega \; , \quad d\mathcal{H} \equiv i_{\vec{\boldsymbol{n}}} \boldsymbol{\eta}$$

If $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ then the conserved charges $\mathcal{Q}(\mathcal{H}_1)$, $\mathcal{Q}(\mathcal{H}_2)$ can be defined by

$$\mathcal{Q}(\mathcal{H}_1) \equiv \int_{\mathcal{H}_1} (n_\mu Z^\mu) d\mathcal{H}_1 \; , \quad \mathcal{Q}(\mathcal{H}_2) \equiv \int_{\mathcal{H}_2} (n_\mu Z^\mu) d\mathcal{H}_2,$$
$$\mathcal{Q}(\mathcal{H}_1) = \mathcal{Q}(\mathcal{H}_2).$$

- Many times one looks for non-negative (resp. non-positive) conserved charges. This requires finding a conserved current Z^μ and $\mathcal{H}_1$, $\mathcal{H}_2$ with special properties.

The role of conserved currents

Standard examples of conserved currents with non-negative conserved charges:

- Symmetric tensor $T_{\mu\nu}$ fulfilling the dominant energy condition and a causal Killing vector ξ^μ. If $\nabla_\mu T^\mu{}_\nu = 0$ and $\mathcal{H}_1$ or $\mathcal{H}_2$ are space-like then

  $Z^\mu \equiv T^\mu{}_\nu \xi^\nu$ is a conserved current with a non-negative conserved charge.

- Bel-Robinson tensor $B_{\mu\nu\alpha\rho}$ and ξ^μ causal Killing vector field. The vector field

  $$Z^\mu \equiv B^\mu{}_{\nu\alpha\rho}\xi^\nu\xi^\alpha\xi^\rho \ ,$$

  is a conserved current in vacuum (use the property $\nabla_\mu B^\mu{}_{\nu\alpha\rho} = 0$) with a non-negative conserved charge if $\mathcal{H}_1$ or $\mathcal{H}_2$ are space-like.

The role of conserved currents

Positivity properties of $T_{\mu\nu}$ and $B_{\mu\nu\alpha\rho}$:

- for any u^μ, v^ν, w^α, z^ρ causal and future directed one has

$$T_{\mu\nu}u^\mu v^\nu \geq 0 \, , \quad B_{\mu\nu\alpha\rho}u^\mu v^\nu w^\alpha z^\rho \geq 0.$$

- u^μ, v^ν, w^α, z^ρ time-like and future-directed

$$T_{\mu\nu}u^\mu v^\nu = 0 \, , \quad B_{\mu\nu\alpha\rho}u^\mu v^\nu w^\alpha z^\rho = 0 \Longleftrightarrow T_{\mu\nu} = 0 \, , \quad B_{\mu\nu\alpha\rho} = 0.$$

Due to the charge conservation and the positivity properties, the vanishing of the conserved charges in a spacelike hypersurface Σ entails the vanishing of $T_{\mu\nu}$ or $B_{\mu\nu\alpha\rho}$ in a neighbourhood of $\Sigma \Rightarrow$ the fields defining these tensors also vanish.

Vacuum spacetimes with a Killing vector

Assume that a vacuum space-time $(\mathcal{M}, g_{\mu\nu})$ admits a Killing vector ξ^μ.

Killing condition: $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$, Killing 2-form: $F_{\mu\nu} \equiv \nabla_{[\mu}\xi_{\nu]} = \nabla_\mu \xi_\nu$.

Self-dual Killing form: $\mathcal{F}_{\mu\nu} \equiv F_{\mu\nu} + \mathrm{i} F^*_{\mu\nu}\ , \quad \mathcal{F}^2 \equiv \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$.

Ernst 1-form: $\sigma_\nu \equiv 2\xi^\mu \mathcal{F}_{\mu\nu}$.

Self-dual Weyl tensor: $\mathcal{W}_{\mu\nu\lambda\rho} \equiv W_{\mu\nu\lambda\rho} + \mathrm{i}\ W^*_{\mu\nu\lambda\rho}$.

σ_μ closed $\Rightarrow \exists$ a local potential σ (Ernst potential) such that $\sigma_\mu = \nabla_\mu \sigma$.

$$\sigma = k + \lambda + 2\ \mathrm{i}\ \omega\ , \quad \mathrm{k} \in \mathbb{C}\ , \quad \lambda \equiv \xi_\mu \xi^\mu\ , \quad \omega \equiv \text{twist potential.}$$

The Mars-Simon tensors

Let $\mathcal{I}_{\mu\nu\lambda\rho} \equiv \frac{1}{4}\left(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda} + \mathrm{i}\eta_{\mu\nu\lambda\rho}\right)$. For any vacuum space-time admitting a Killing vector we define the family of Mars-Simon tensors

$$\mathcal{S}_{\mu\nu\lambda\rho} \equiv \mathcal{W}_{\mu\nu\lambda\rho} + \frac{6}{\sigma}\left(\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\rho} - \frac{\mathcal{F}^2}{3}\mathcal{I}_{\mu\nu\lambda\rho}\right) , \quad \sigma \neq 0.$$

The Mars-Simon tensors are Weyl candidates:

$$\mathcal{S}_{[\mu\nu]\alpha\beta} = \mathcal{S}_{\mu\nu\alpha\beta} , \quad \mathcal{S}_{\mu\nu\alpha\beta} = \mathcal{S}_{\alpha\beta\mu\nu} , \quad \mathcal{S}_{[\mu\nu\alpha]\beta} = 0 , \quad \mathcal{S}^{\mu}{}_{\mu\alpha\beta} = 0 ,$$

and they are self-dual:

$$\mathrm{i}\, \mathcal{S}^{*}_{\mu\nu\alpha\beta} = \mathcal{S}_{\mu\nu\alpha\beta}.$$

For any point we can choose the Ernst potential in such a way that $\sigma \neq 0$. Therefore the Mars-Simon tensors can be defined (at least locally) for any vacuum space-time.

A local invariant characterisation of the Kerr solution

The Mars-Simon tensor is used in the following important result.

Theorem (Marc Mars (2000))

Let $(\mathcal{M}, g_{\mu\nu})$ *be a smooth non-trivial vacuum solution having a Killing vector* $\vec{\xi}$ *and assume that it fulfills the following conditions*

- $(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) \neq 0$.
- *There is a choice of the Ernst potential* σ *for which*

$$\mathcal{S}_{\mu\nu\rho\lambda} = 0\,, \quad \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{\sigma^4}{4M^2} = 0\,, \quad M \in \mathbb{R}\setminus\{0\}\,, \quad \mathrm{Re}(\sigma) - \lambda > 0\,,$$

- *There is at least a point* q *such that the Killing vector* $\vec{\xi}|_q$ *does not lie in the 2-plane orthogonal to the 2-plane spanned by the two independent null eigenvectors of* $\mathcal{F}_{\mu\nu}|_q$.

Under the previous assumptions the space-time is locally isometric to the Kerr solution.

Our result

Introduce the super-energy tensor of a Mars-Simon tensor:

$$T_{\alpha\beta\gamma\delta} \equiv \mathcal{S}_{\alpha}{}^{\mu\nu}{}_{\beta}\bar{\mathcal{S}}_{\gamma\mu\nu\delta}.$$

$T_{\alpha\beta\gamma\delta}$ has the same positivity and algebraic properties as the Bel-Robinson tensor.

Theorem

For any vacuum solution of the Einstein's field equations $(\mathcal{M}, g_{\mu\nu})$ *admitting a Killing vector field, consider an open subset* $\mathcal{U} \subset \mathcal{M}$ *and a choice of the Ernst potential which is differentiable and non-vanishing. Define the tensor* $T_{\alpha\beta\gamma\delta}$ *as explained above. Then the current*

$$P^{\alpha} \equiv \frac{1}{|\sigma|^6} T^{\alpha}{}_{\beta\gamma\delta}\xi^{\beta}\xi^{\gamma}\xi^{\delta} \ ,$$

is conserved in $\mathcal{U}$

$$\nabla_{\alpha}P^{\alpha} = 0.$$

Remarks

- Use the freedom we have to define the Ernst potential to make a choice which does not vanish at a given point

$$\sigma \to \sigma + k\,, \quad k \in \mathbb{C}.$$

  Therefore the current $\vec{\boldsymbol{P}}$ can be define at least locally for any vacuum space-time.
- The current $\vec{\boldsymbol{P}}$ is in fact a family which depends on a complex constant k. For example for the Schwarzschild space-time and using Schwarzschild coordinates:

$$\vec{\boldsymbol{\xi}} = \frac{\partial}{\partial t} \Rightarrow \vec{\boldsymbol{P}} = \frac{6M^2|k|^2 r(r-2M)}{|kr-2M|^8}\frac{\partial}{\partial t}\,, \quad k \in \mathbb{C}.$$

- If the Killing vector $\vec{\boldsymbol{\xi}}$ is causal then the generalised dominant energy property of $T_{\mu\nu\alpha\rho}$ implies that the conserved current $\vec{\boldsymbol{P}}$ is causal too. Moreover under this assumption

$$\vec{\boldsymbol{P}} = 0 \Longleftrightarrow T_{\mu\nu\alpha\rho} = 0 \Longleftrightarrow \mathcal{S}_{\mu\nu\alpha\rho} = 0.$$

A conserved charge characterising the Kerr solution

Theorem (Kerr conserved charge)

Let $(\mathcal{M}, g_{\mu\nu})$ *be a vacuum stationary solution of the Einstein's field equations and assume further that for a given embedded space-like hypersurface* $\Sigma \subset \mathcal{M}$, $(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})|_{\Sigma} \neq 0$ *and the Ernst potential is chosen in such a way that it fulfills*

$$\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{\sigma^4}{4M^2}\right)\Big|_{\Sigma} = 0\,, \quad M \in \mathbb{R}\setminus\{0\}\,, \quad (Re(\sigma) - \lambda)\,|_{\Sigma} > 0.$$

Use the stationary Killing vector and the Ernst potential to define the conserved current $\vec{\boldsymbol{P}}$*. Then the scalar* $\mathcal{Q}(\Sigma)$ *defined by*

$$\mathcal{Q}(\Sigma) \equiv \int_{\Sigma} P^{\mu} n_{\mu} d\Sigma\,,$$

is non-negative, it vanishes if and only if Σ *can be isometrically embedded within an open subset of the Kerr solution and it is a conserved charge.*

Open issues

- Attempt to construct similar conserved currents for vacuum with non-vanishing cosmological constant. Use a suitable generalisation of the Mars-Simon tensor family (Mars & Senovilla 2015)

$$\mathcal{S}_{\mu\nu\lambda\rho} \equiv \mathcal{W}_{\mu\nu\lambda\rho} + Q\left(\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\rho} - \frac{\mathcal{F}^2}{3}\mathcal{I}_{\mu\nu\lambda\rho}\right), \quad Q \in C^\infty(\mathcal{M}, \mathbb{C}).$$

- One of the hypotheses of our theorem about the Kerr conserved charge, is that the hypersurface is spacelike. One could attempt to generalise it for hypersurfaces with a mixed causal character (spacelike-null).
- Render the Kerr conserved charge in terms of initial data for the vacuum Einstein equations or quantities intrinsic to the hypersurface Σ. Use the notion of Killing initial data.