Quantum Logic

Carlos Fitas

CMAT

Q DAYS

April 11, 2019

Introduction

Quantum logic was created by Birkhoff and von Neumann in the thirties as a extrapolation from the algebraic structure of the set of closed subspaces of a Hilbert space.

Classical particle mechanics

Let σ be a classical physical system (let's say one classical particle).

One can associate to σ, as a mathematical representation, a **phase-space** P.

P is the set of all 6−tuples $(x_1 \dots x_6)$ of real numbers:

- x_1, x_2, x_3 representing three *position* coordinates;
- x_4, x_5, x_6 representing three *momentum* coordinates.

Any $p \in P$ represents a **pure state**.

Classical particle mechanics

In this framework, it's natural to assume that the power-set $\mathcal{P}(P)$ of P represents all of the possible **properties** of the pure states.

For instance, the property "no momentum" is simply the set

$$\{(x_1, x_2, x_3, 0, 0, 0) : x_1, x_2, x_3 \in \mathbb{R}\}$$

Classical particle mechanics

Using terminology from logic we may say that any property $X \in \mathcal{P}(P)$ represents a **proposition** which may be **true** or **false** for any given pure state p:

- X is true if $p \in X$;
- X is false if $p \in P \setminus X$.

For instance, the property "no momentum"

$$\{(x_1, x_2, x_3, 0, 0, 0) : x_1, x_2, x_3 \in \mathbb{R}\}$$

seen as a proposition is

- true for the pure state $(2, 3, 6, 0, 0, 0)$;
- false for the pure state $(2, 3, 6, 0, 0, 1)$.

Classical particle mechanics

Since $\mathcal{P}(P)$ has a Boolean structure, it's governed by classical logic, with the set-theoretical operations seen as logical connectives:

complement	$\sim$
intersection	$\wedge$
union	$\vee$

Quantum theory

In the standard formalism of quantum theory:

- the role of the phase-space P is played by a Hilbert space $\mathcal{H}$;
- the pure states of a system are the unit vectors in $\mathcal{H}$;

We are only interested in properties that can in principle be tested by a measurement. These are called **testable properties**. In our Hilbert space $\mathcal{H}$, the set of testable properties is the set $\mathcal{C}(\mathcal{H})$ of closed linear subspaces of $\mathcal{H}$.

However, unlike $\mathcal{P}(P)$, $\mathcal{C}(\mathcal{H})$ is not closed under the set-theoretical operations.

Consequently we cannot define a Boolean structure on $\mathcal{C}(\mathcal{H})$, using the set-theoretical operations.

Quantum theory

Nevertheless, we will see that $\mathcal{C}(\mathcal{H})$ can be extended, in a natural way, to a certain "quasi-Boolean" algebraic structure.

These structures are called **ortholattices**.

Ortholattices

Definition (Ortholattice)

An **ortholattice** is a structure $\mathcal{O} = (\mathcal{O}, \leq, \sqcap, \sqcup, \neg, \bot, \top)$, where

- $(\mathcal{O}, \leq, \sqcap, \sqcup, \bot, \top)$ is lattice with **maximum** $(\top)$ and **minimum** $(\bot)$;
- $\neg$ is a $1-$ary operation, called **orthocomplement**, satisfying:
  - $\neg\neg A = A$,
  - $A \leq B \Rightarrow \neg B \leq \neg A$,
  - $A \sqcap \neg A = 0$,
  - $A \sqcup \neg A = 1$,

  for all $A, B \in \mathcal{O}$.

The usual De Morgan's laws are valid:

$$\neg\bot = \top \qquad \neg(A \sqcup B) = \neg B \sqcap \neg A$$
$$\neg\top = \bot \qquad \neg(A \sqcap B) = \neg B \sqcup \neg A$$

Ortholattice of closed linear subspaces

It's easy to check that $(\mathcal{C}(\mathcal{H}), \subseteq, \cap, +, \neg, \mathbf{0}, \mathcal{H})$ is a ortholattice, where:

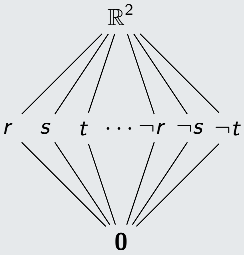
- $A \cap B$ is the set theoretical intersection;
- $A + B := \langle A \cup B \rangle$, the linear subspace generated by the set-theoretical union $A \cup B$;
- $\neg A$ is the subspace orthogonal to A, $\neg A = \{y \in \mathcal{H} \mid \forall x \in A \ \langle x, y \rangle = 0\}$.

Example

Example

Consider the ortholattice $(\mathcal{C}(\mathbb{R}^2), \subseteq, \cap, +, \neg, \mathbf{0}, \mathbb{R}^2)$.

In this case $\mathcal{C}(\mathbb{R}^2)$ is simply the set containing all the straight lines through the origin, the whole plane ($\mathbb{R}^2$) and the origin ($\mathbf{0}$):



where r, s, t are straight lines through the origin.

Violation of meta-*tertium non datur*

Remember that each $A \in \mathcal{C}(\mathcal{H})$ represents a proposition which is, for a given pure state p,

- true if $p \in A$;
- false if $p \in \neg A$.

It's perfectly possible that a proposition is neither true nor false.

In other words, we have a violation of the meta-theoretical *tertium non datur*.

At the same time $A + \neg A$ is true for any pure state p and any proposition A.

Which means that the theoretical *tertium non datur* holds!

Violation of meta-*tertium non datur*

This can be explained by the fact that, the truth of a disjunction does not imply the truth of at least one member.

While this result is counter-intuitive it mirrors some "logical anomalies" of quantum mechanics.

Consider the famous "two-slit experiment". In this physical experiment we have a certain particle p and we know that:

"p has gone trough slit A" or "p has gone trough slit B"

Yet, we can neither maintain that it is true that,

"p has gone trough slit A"

nor that it is true that

"p has gone trough slit B"

Orthomodularity

While we don't generally have distributivity in ortholattices we do have a weak form of modularity for the class of ortholattices corresponding to $\mathcal{C}(\mathcal{H})$.

Recall that a lattice is modular if the following identity holds

$$A \leq B \Rightarrow (A \vee C) \wedge B \leq A \vee (C \wedge B)$$

Orthomodularity only requires this identity for the special case $C = \neg A$:

Definition (Orthomodularity)

An ortholattice $\mathcal{O}$ is **orthomodular** if

$$A \leq B \Rightarrow B \leq (A \sqcup (\neg A \sqcap B))$$

holds for any $A, B \in \mathcal{O}$.

Quantum Logic

There are two variants of quantum logic:

- Orthomodular quantum logic (**OQL**)
  - It's associated to the class of orthomodular ortholattices.
  - It's a more faithful representation of the formalism of quantum theory.
- Minimal quantum logic (**MQL**)
  - It's associated to the class of ortholattices.
  - Has much better logical proprieties than orthomodular quantum logic.

Algebraic Semantics

There are well-known algebraic semantics for classic, intuitionistic, and many other logics.

For instance, classic logic and intuitionistic logic can be interpreted with boolean lattices and Heyting lattices, respectively.

We can, analogously, interpret **OQL** and **MQL** with orthomodular lattices.

In sum:

Logic	Algebra
Classic Logic	Boolean lattices
Intuitionistic Logic	Heyting lattices
Quantum Logic	Orthomodular lattices

Syntax

The formulas for **OQL** and **MQL** are built using only two connectives:

$$A ::= x \mid A \sqcap A \mid \neg A,$$

where x ranges over elements of a given countable set $\mathcal{X}$ of **variables**.

We now use $\sqcap$ and $\neg$ to define the following:

$$A \sqcup B := \neg(\neg A \sqcap \neg B)$$
$$\bot := \neg x \sqcap x$$
$$\top := \neg \bot$$

Algebraic realization

Definition (Algebraic realization)

An **algebraic realization** for **MQL** (resp. **OQL**) is a pair $\mathcal{R} = (\mathcal{O}, v)$ where

- $\mathcal{O}$ is an ortholattice (resp. orthomodular ortholattice);
- v is a **valuation-function** which associates with any formula A an element of $\mathcal{O}$ and satisfies the following conditions:
  - $v(\neg A) = \neg v(A)$,
  - $v(A \sqcap B) = v(A) \sqcap v(B)$.

Logical Truth

Definition (Logical Truth)

Let A be a formula.

A is **true** in a algebraic realization $\mathcal{R} = (\mathcal{O}, v)$ if $v(A) = \top$. In that case we write $\models_{\mathcal{R}} A$.

A is a **logical truth** of **MQL** (resp. **OQL**) if A is true for any algebraic realization. In that case we write $\models_{\mathbf{MQL}} A$ $\left(\text{resp. } \models_{\mathbf{OQL}} A\right)$.

Logical Consequence

Definition (Logical Consequence)

Let Γ be a set of formulas.

A is a **logical consequence** of Γ if, for any algebraic realization $\mathcal{R} = (\mathcal{O}, v)$, any $o \in \mathcal{O}$

$$\text{any } B \in \Gamma, \quad o \leq v(B) \Rightarrow o \leq v(A).$$

In that case we write $\Gamma \models_{\mathbf{MQL}} A$ $\left(\text{resp. } \Gamma \models_{\mathbf{OQL}} A\right)$.

We will write $B \models_{\mathbf{-QL}} A$ instead of $\{B\} \models_{\mathbf{-QL}} A$.

It's easy to check that $B \models_{\mathbf{-QL}} A \Leftrightarrow v(B) \leq v(A)$ for any algebraic realization.

Axiomatization of Minimal Quantum Logic

We will now axiomatize the consequence relation $\vdash$ of minimal quantum logic.

Naturally, we want

$$A \vdash B \text{ to be derivable if and only if } A \models_{\mathbf{MQL}} B.$$

Equivalently, we want

$$A \vdash B \text{ to be derivable if and only if } v(A) \leq v(B),$$

for any algebraic realization.

Goldblatt's axiomatization [1974]

$$\frac{}{A \vdash A}\,ax \qquad \frac{A \vdash B \qquad B \vdash C}{A \vdash C}\,cut$$

$$\frac{}{A \sqcap B \vdash A}\,\sqcap_1 L \qquad \frac{}{A \sqcap B \vdash B}\,\sqcap_2 L \qquad \frac{C \vdash A \qquad C \vdash B}{C \vdash A \sqcap B}\,\sqcap R \qquad \frac{}{C \vdash \top}\,\top R$$

$$\frac{}{A \vdash A \sqcup B}\,\sqcup_1 R \qquad \frac{}{B \vdash A \sqcup B}\,\sqcup_2 R \qquad \frac{A \vdash C \qquad B \vdash C}{A \sqcup B \vdash C}\,\sqcup L \qquad \frac{}{\bot \vdash C}\,\bot L$$

$$\frac{A \vdash B}{\neg B \vdash \neg A}\,\neg \qquad \frac{}{A \vdash \neg\neg A}\,\neg\neg R \qquad \frac{}{\neg\neg A \vdash A}\,\neg\neg L \qquad \frac{}{\top \vdash A \sqcup \neg A}\,tnd$$

Goldblatt's axiomatization [1974]

To obtain a calculus for **OQL** we simply have to add another rule:

$$\dfrac{}{A \sqcap (\neg A \sqcup (A \sqcap B)) \vdash B}\ om$$

A problem of Goldblatt's axiomatization

It's not possible to eliminate the cut rule:

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}\, cut$$

This makes studying the derivability of $A \vdash C$ very complicated, since we may need to invent some B seemingly unrelated to A and C.

On the other hand, cut-free systems usually satisfy the **sub-formula property**: every formula appearing in a derivation of $A \vdash C$ is a sub-formula of A or C.

Oliver Laurent's axiomatization [2017]

$$\frac{}{\vdash \neg A, A}\ ax \qquad \frac{}{\vdash \top, C}\ \top \qquad \frac{\vdash A}{\vdash A, B}\ w$$

$$\frac{\vdash A, C}{\vdash A \sqcup B, C}\ \sqcup_1 \qquad \frac{\vdash B, C}{\vdash A \sqcup B, C}\ \sqcup_2 \qquad \frac{\vdash A, C \qquad \vdash B, C}{\vdash A \sqcap B, C}\ \sqcap$$

For this calculus we have that

$$\vdash \neg A, B \text{ is derivable if and only if } v(A) \leq v(B),$$

for any algebraic realization.

Implication in Classic Logic

In classical logic we can define an implication connective using the classic negation $\sim$ and classic disjunction $\vee$:

$$A \rightarrow B := \sim A \vee B$$

This implication is generally called **material implication**.

Implication in Classic Logic

In the classical calculus:

- modus ponens holds:

$$A \wedge (A \to B) \vdash B$$

- deduction theorem is provable:

$$\Gamma, A \vdash B \Rightarrow \Gamma \vdash A \to B$$

The deduction theorem together with modus ponens can be stated simply as the following **implicative rule**:

$$A \wedge B \vdash C \Leftrightarrow B \vdash A \to C$$

The Problem

It is known that any logic with a binary connective satisfying the implicative rule is distributive.

Hence it comes as no surprise that the implicative rule cannot be encountered in quantum logic because of the failure of distributivity.

This is the so called **implication problem**.

At the end of the seminal paper "The Logic of Quantum Mechanics":

> "*Our conclusion agrees perhaps more with those critiques of logic, which find most objectionable the assumption that* $a' \cup b = \top$ *implies* $a \subset b$."
>
> G. Birkhoff and J. von Neumann

Implication in Quantum Logic

It is natural to wonder if possible to define some other kind of implication in quantum logic.

Let us first assume that any implication operation should satisfy the **law of entailment**:

$$A \vdash B \Leftrightarrow \vdash A \to B$$

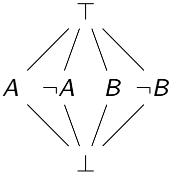
Under this assumption, there are exactly five possible definitions for a binary implication (in terms of $\neg$ and $\sqcap$):

- $A \to_1 B := (\neg A \sqcap B) \sqcup (\neg A \sqcap \neg B) \sqcup (A \sqcap (\neg A \sqcup B))$
- $A \to_2 B := (\neg A \sqcap B) \sqcup (A \sqcap B) \sqcup ((\neg A \sqcup B) \sqcap \neg B)$
- $A \to_3 B := \neg A \sqcup (A \sqcap B)$
- $A \to_4 B := B \sqcup (\neg A \sqcap \neg B)$
- $A \to_5 B := (\neg A \sqcap B) \sqcup (A \sqcap B) \sqcup (\neg A \sqcap \neg B)$

Implication in Quantum Logic

Note that the classical material implication is not one these five implication operations as it violates the law of entailment.

Consider the following orthomodular lattice:



It's clear that we have $\vdash \neg A \sqcup B$ but not $A \vdash B$.

Sasaki Hook

One of these five implications operations is particularly interesting:

$$A \to_3 B := \neg A \sqcup (A \sqcap B),$$

the so called **Sasaki hook**.

This implication is “better” than the other candidates because it perfectly matches classical implication if the elements are **compatible**:

$$A \to_3 B = \neg A \sqcup B \text{ if } A = (A \sqcap B) \sqcup (A \sqcap \neg B)$$

Sasaki Hook

In the case of **OQL**, the Sasaki Hook also satisfies *modus ponens*:

$$A \sqcap (A \rightarrow_3 B) \vdash B$$

which is exactly the orthomodular rule:

$$\frac{}{A \sqcap (\neg A \sqcup (A \sqcap B)) \vdash B}\ om$$

Sasaki Hook - Non-properties

It's worth stressing that the Sasaki hook **lacks** many properties typically associated with implication operations:

- transitivity

$$A \to_3 B \sqcap B \to_3 C \not\vdash A \to_3 C$$

- weakening

$$A \to_3 C \not\vdash (A \sqcap B) \to_3 C$$

- contraposition

$$A \to_3 B \not\vdash \neg B \to_3 \neg A$$

References

- Garrett Birkhoff & John Von Neumann
  The Logic of Quantum Mechanics.
  The Annals of Mathematics, 2nd Series, Vol. 37, No. 4, 1936
- Dalla Chiara
  Quantum Logic.
  Handbook of Philosophical Logic, Vol. 166, Springer, 1986
- Robert Goldblatt
  Semantics of Orthologic.
  Journal of Philosophical Logic Vol. 3, No. 1/2, 2010
- Herman, Marsden & Piziak
  Implication connectives in orthomodular lattices.
  Notre Dame J. Formal Logic, Vol. 16, 1975
- Gudrun Kalmbach
  Orthomodular Lattices
  Academic Press, London, 1983
- Oliver Laurent
  Focusing in Orthologic.
  Logical Methods in Computer Science Vol. 13, 2017
- Sonja Smets
  Logic and Quantum Physics.
  Journal of the Indian Council of Philosophical Research, Vol. 27, 2010