## A survey of Focal Decomposition

The concept of focal decomposition was introduced in [1] in the context of the 2–point boundary value problem

$$(1) \qquad \ddot{x} = f(t, x, \dot{x}) \;, \qquad x(t_1) = x_1 \;, \qquad x(t_2) = x_2 \;,$$

the simplest and oldest of all boundary value problems.

Consider now the space $\mathbb{R}^4$ $(t_1, x_1 ; t_2, x_2)$ of all pair of points in $\mathbb{R}^2$. If $i$ is the number of solutions of the problem (1), a non negative integer or $\infty$ , we say that $i$ is the index of the point $(t_1, x_1 ; t_2, x_2)$ . Call $\Sigma_i$ the totality of points of $\mathbb{R}^4$ $(t_1, x_1 ; t_2, x_2)$ with index $i$ . These sets then determine a partition

$$\mathbb{R}^4 = \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_\infty \;,$$

the focal decomposition associated to the 2–point boundary value problem (1).

This focal decomposition is then the object of study.

From the work of Peixoto and Thom [2] the possibility of a general, analytic theory became clear. We have then an existence theorem which says that under a certain properness condition expressed in terms of the solutions of (1), the sets $\Sigma_i$ are the unions of strata of an analytical Whitney stratification of $(t_1, x_1 ; t_2, x_2)$ – space $\mathbb{R}^4$ minus the diagonal $t_1 = t_2$ . See [3].

Afterwards Kupka and Peixoto [4] extended the concept of focal decomposition in the context of an analytical Riemannian manifold.

In this context natural relationships of focal decompositions with such objects as Brillouin zones, semi-classical quantization and Diophantine equations were discovered.

### References


[1] *M. M. Peixoto*, On Endpoint Boundary Value Problems, J. Diff. Equ. **44** (1982), 273-280.

[2] *M. M. Peixoto* and *R. Thom*, Le point de vue énumeratif dans les problems aux limites pour les équations différentielles ordinaires I, II, C.R. Acad. Sci. **303** (1986), 629-632, 693-698 ; Erratum **307** (1988), 197-198.

[3] *M. M. Peixoto* and *A. R. da Silva*, Focal decomposition and some results of S. Bernstein on the 2–point boundary value problem, J. London Math. Soc. (2) **60** (1999), 517-547.

[4] *I. Kupka* and *M. M. Peixoto*, On the Enumerative Geometry of Geodesics, From Topology to Computation, Proceedings of the Smalefest, Springer-Verlag (1993), 243-253.