Three Lectures on Hybrid Logic

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- We solved this by introducing nominals—propositional symbols i, j, k constrained in their interpretation symbols to be true at a unique world in any model. We also introduced it the @-operator.
- We showed that our new hybrid tools both dealt with the expressive shortcomings and had inherited modal beauty. And, by lifting the van Benthem Characterization Theorem to cover them, we concluded that we were still doing honest-to-goodness, down-home, good-'ol modal logic....

Today: Hybrid deduction

Let's continue with an example-driven introduction to hybrid deduction. We concentrate on tableau systems. We shall:

- Discuss the goals and problems of orthodox modal deduction.
- Present a hybrid tableau system for reasoning about arbitrary models.
- Show how this can be extended to hybrid tableau systems for special classes of models.
- Round off by discussing further themes in hybrid deduction, including their implementation.

Different models, different logics

Key fact about modal logic: when you work with different kinds of models (graphs) the logic typically changes. For example:

- $\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$ is valid on all models: it's part of the basic, universally applicable, logic.
- But $\Diamond\Diamond p \rightarrow \Diamond p$ is only valid on transitive graphs. It's not part of the basic logic, rather it's part of the special (stronger) logic that we need to use when working with transitive models.

Modal deduction should be general

- Quite rightly, modal logicians have insisted on developing proof methods which are general — that is, which can be easily adapted to cope with the logics of many kinds of models (transitive, reflexive, symmetric, dense, and so on).
- They achieve this goal by making use of Hilbert-style systems (that is, axiomatic systems).
- There is a basic axiomatic systems (called **K**) for dealing with arbitrary models.
- To deal with special classes of models, further axioms are added to **K**. For example, adding $\Diamond\Diamond p \to \Diamond p$ as an axiom gives us the logic of transitive frames.

Generality clashes with easy of use

- Unfortunately, Hilbert systems are hard to use and completely unsuitable for computational implementation.
- For ease of use we want (say) natural deduction systems or tableau systems. For computational implementation we want (say) resolution systems or tableau systems.
- But it is hard to develop tableau, or natural deduction, or resolution in a general way in orthodox modal logic.
- Why is this?

Getting behind the diamonds

- The difficulty is extracting information from under the scope of diamonds.
- That is, given $\Diamond\varphi$, how do we lay hands on φ? And given $\neg\Box\varphi$ (that is, $\Diamond\neg\varphi$), how do we lay hands on $\neg\varphi$?
- In first order logic, the analogous problem is trivial. There is a simple rule for stripping away existential quantifiers: from $\exists x\varphi$ we conclude $\varphi[x \leftarrow a]$ for some brand new constant a (this rule is usually called Existential Elimination).
- But in orthodox modal logic there is no simple way of stripping off the diamonds.

Hybrid deduction

- Hybrid deduction is based on a simple observation: it's easy to get at the information under the scope of diamonds — for there is an natural way of stripping the diamonds away.
- We shall explore this idea in the setting of tableau — but it can (and has been) used in a variety of proof styles, including resolution and natural deduction.
- Moreover, once the tableau system for reasoning about arbitrary models has been defined, it is straightforward to extend it to to cover the logics of special classes of models. That is, hybridization enables us to achieve the traditional modal goal of generality without resorting to Hilbert-systems.

Moreover...

Hybrid reasoning is arguably quite natural.

In what follows we shall sometimes give an informal proof before we give the tableau proof. As we shall see, our tableau proofs mimic the informal reasoning fairly closely.

Hip and cute

Consider the following statement:

> *If everyone you hate is hip, and someone you hate is cute, then someone you hate is both hip and cute.*

We can represent it as follows:

$$[\textsc{hate}]\,\mathrm{hip} \wedge \langle\textsc{hate}\rangle\,\mathrm{cute} \to \langle\textsc{hate}\rangle\,(\mathrm{hip} \wedge \mathrm{cute})$$

This is a valid statement, and it's validity is easy to establish informally. . .

Informal argument

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- Then everyone you hate is hip, and someone you hate is cute. However no one you hate is both hip and cute.

Informal argument

- Suppose "If everyone you hate is hip, and someone you hate is cute, then someone you hate is both hip and cute" is not true.
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- So there is someone that you hate—let's call him Jim—who is cute.

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- Then everyone you hate is hip, and someone you hate is cute. However no one you hate is both hip and cute.
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- But as Jim is someone you hate, he will be hip as well as cute (for everyone you hate is hip).

Informal argument

- Suppose "If everyone you hate is hip, and someone you hate is cute, then someone you hate is both hip and cute" is not true.
- Then everyone you hate is hip, and someone you hate is cute. However no one you hate is both hip and cute.
- So there is someone that you hate—let's call him Jim—who is cute.
- But as Jim is someone you hate, he will be hip as well as cute (for everyone you hate is hip).
- But Jim can't be both hip and cute (for no one you hate is both hip and cute). Contradiction!. So the original statement was true after all.

$$[\textsc{hate}]\,\mathrm{hip} \wedge \langle \textsc{hate} \rangle\,\mathrm{cute} \to \langle \textsc{hate} \rangle\,(\mathrm{hip} \wedge \mathrm{cute})$$

$[\text{HATE}]\,\text{hip} \wedge \langle\text{HATE}\rangle\,\text{cute} \rightarrow \langle\text{HATE}\rangle\,(\text{hip} \wedge \text{cute})$

1 $\neg @_i([\text{HATE}]\,\text{hip} \wedge \langle\text{HATE}\rangle\,\text{cute} \rightarrow \langle\text{HATE}\rangle\,(\text{hip} \wedge \text{cute}))$

$[\text{HATE}]\,\text{hip} \wedge \langle\text{HATE}\rangle\,\text{cute} \to \langle\text{HATE}\rangle\,(\text{hip} \wedge \text{cute})$

1	$\neg @_i([\text{HATE}]\,\text{hip} \wedge \langle\text{HATE}\rangle\,\text{cute} \to \langle\text{HATE}\rangle\,(\text{hip} \wedge \text{cute}))$
2	$@_i([\text{HATE}]\,\text{hip} \wedge \langle\text{HATE}\rangle\,\text{cute})$
$2'$	$\neg @_i\langle\text{HATE}\rangle\,(\text{hip} \wedge \text{cute})$

$[\mathrm{HATE}]\,\mathrm{hip} \wedge \langle\mathrm{HATE}\rangle\,\mathrm{cute} \to \langle\mathrm{HATE}\rangle\,(\mathrm{hip} \wedge \mathrm{cute})$

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2	$@_i([\mathrm{HATE}]\,\mathrm{hip} \wedge \langle\mathrm{HATE}\rangle\,\mathrm{cute})$
$2'$	$\neg @_i\langle\mathrm{HATE}\rangle\,(\mathrm{hip} \wedge \mathrm{cute})$
3	$@_i[\mathrm{HATE}]\,\mathrm{hip}$
$3'$	$@_i\langle\mathrm{HATE}\rangle\,\mathrm{cute}$

$[\mathrm{HATE}]\,\mathrm{hip} \wedge \langle\mathrm{HATE}\rangle\,\mathrm{cute} \to \langle\mathrm{HATE}\rangle\,(\mathrm{hip} \wedge \mathrm{cute})$

1	$\neg @_i([\mathrm{HATE}]\,\mathrm{hip} \wedge \langle\mathrm{HATE}\rangle\,\mathrm{cute} \to \langle\mathrm{HATE}\rangle\,(\mathrm{hip} \wedge \mathrm{cute}))$
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$2'$	$\neg @_i\langle\mathrm{HATE}\rangle\,(\mathrm{hip} \wedge \mathrm{cute})$
3	$@_i[\mathrm{HATE}]\,\mathrm{hip}$
$3'$	$@_i\langle\mathrm{HATE}\rangle\,\mathrm{cute}$
4	$@_i\langle\mathrm{HATE}\rangle\,\mathit{jim}$
$4'$	$@_{\mathit{jim}}\mathrm{cute}$

$[\mathrm{HATE}]\,\mathrm{hip} \wedge \langle\mathrm{HATE}\rangle\,\mathrm{cute} \to \langle\mathrm{HATE}\rangle\,(\mathrm{hip} \wedge \mathrm{cute})$

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2	$@_i([\mathrm{HATE}]\,\mathrm{hip} \wedge \langle\mathrm{HATE}\rangle\,\mathrm{cute})$
$2'$	$\neg @_i\langle\mathrm{HATE}\rangle\,(\mathrm{hip} \wedge \mathrm{cute})$
3	$@_i[\mathrm{HATE}]\,\mathrm{hip}$
$3'$	$@_i\langle\mathrm{HATE}\rangle\,\mathrm{cute}$
4	$@_i\langle\mathrm{HATE}\rangle\,jim$
$4'$	$@_{jim}\mathrm{cute}$
5	$@_{jim}\mathrm{hip}$

$[\mathrm{HATE}]\,\mathrm{hip} \wedge \langle \mathrm{HATE} \rangle\, \mathrm{cute} \to \langle \mathrm{HATE} \rangle\, (\mathrm{hip} \wedge \mathrm{cute})$

1	$\neg @_i([\mathrm{HATE}]\,\mathrm{hip} \wedge \langle \mathrm{HATE} \rangle\, \mathrm{cute} \to \langle \mathrm{HATE} \rangle\, (\mathrm{hip} \wedge \mathrm{cute}))$
2	$@_i([\mathrm{HATE}]\,\mathrm{hip} \wedge \langle \mathrm{HATE} \rangle\, \mathrm{cute})$
$2'$	$\neg @_i \langle \mathrm{HATE} \rangle\, (\mathrm{hip} \wedge \mathrm{cute})$
3	$@_i [\mathrm{HATE}]\,\mathrm{hip}$
$3'$	$@_i \langle \mathrm{HATE} \rangle\, \mathrm{cute}$
4	$@_i \langle \mathrm{HATE} \rangle\, jim$
$4'$	$@_{jim} \mathrm{cute}$
5	$@_{jim} \mathrm{hip}$
6	$\neg @_{jim} (\mathrm{hip} \wedge \mathrm{cute})$

$[\mathrm{HATE}]\,\mathrm{hip} \wedge \langle \mathrm{HATE} \rangle\, \mathrm{cute} \to \langle \mathrm{HATE} \rangle\, (\mathrm{hip} \wedge \mathrm{cute})$

1		$\neg @_i([\mathrm{HATE}]\,\mathrm{hip} \wedge \langle \mathrm{HATE} \rangle\, \mathrm{cute} \to \langle \mathrm{HATE} \rangle\, (\mathrm{hip} \wedge \mathrm{cute}))$	
2		$@_i([\mathrm{HATE}]\,\mathrm{hip} \wedge \langle \mathrm{HATE} \rangle\, \mathrm{cute})$	
$2'$		$\neg @_i \langle \mathrm{HATE} \rangle\, (\mathrm{hip} \wedge \mathrm{cute})$	
3		$@_i [\mathrm{HATE}]\,\mathrm{hip}$	
$3'$		$@_i \langle \mathrm{HATE} \rangle\, \mathrm{cute}$	
4		$@_i \langle \mathrm{HATE} \rangle\, \mathit{jim}$	
$4'$		$@_{\mathit{jim}} \mathrm{cute}$	
5		$@_{\mathit{jim}} \mathrm{hip}$	
6		$\neg @_{\mathit{jim}} (\mathrm{hip} \wedge \mathrm{cute})$	
7	$\neg @_{\mathit{jim}} \mathrm{hip}$		$\neg @_{\mathit{jim}} \mathrm{cute}$
	$\perp_{5,7}$		$\perp_{4',7}$

Internalizing Labelled Deduction

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$\neg$ rules $\qquad \dfrac{@_i \neg\varphi}{\neg @_i \varphi} \qquad \dfrac{\neg @_i \neg\varphi}{@_i \varphi}$

Internalizing Labelled Deduction

$\neg$ rules	$\dfrac{@_i\neg\varphi}{\neg @_i\varphi}$	$\dfrac{\neg @_i\neg\varphi}{@_i\varphi}$
$\wedge$ rules	$\dfrac{@_i(\varphi \wedge \psi)}{\begin{array}{c} @_i\varphi \\ @_i\psi \end{array}}$	$\dfrac{\neg @_i(\varphi \wedge \psi)}{\neg @_i\varphi \mid \neg @_i\psi}$

Internalizing Labelled Deduction

$\neg$ rules

$$\frac{@_i \neg\varphi}{\neg @_i \varphi} \qquad \frac{\neg @_i \neg\varphi}{@_i \varphi}$$

$\wedge$ rules

$$\frac{@_i(\varphi \wedge \psi)}{\begin{array}{c} @_i\varphi \\ @_i\psi \end{array}} \qquad \frac{\neg @_i(\varphi \wedge \psi)}{\neg @_i\varphi \mid \neg @_i\psi}$$

$@$ rules

$$\frac{@_i @_j \varphi}{@_j \varphi} \qquad \frac{\neg @_i @_j \varphi}{\neg @_j \varphi}$$

Extracting information from modal contexts

In the statement of these rules we write j to indicate a nominal new to the branch where the rule is being applied.

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$\Diamond$ rules

$$\frac{@_i \langle \mathrm{R} \rangle \varphi}{\begin{array}{l} @_i \langle \mathrm{R} \rangle j \\ @_j \varphi \end{array}} \qquad\qquad \frac{\neg @_i \langle \mathrm{R} \rangle \varphi \quad @_i \langle \mathrm{R} \rangle k}{\neg @_k \varphi}$$

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$\Diamond$ rules

$$\frac{@_i \langle \mathrm{R} \rangle \varphi}{\begin{array}{l} @_i \langle \mathrm{R} \rangle j \\ @_j \varphi \end{array}} \qquad \frac{\neg @_i \langle \mathrm{R} \rangle \varphi \quad @_i \langle \mathrm{R} \rangle k}{\neg @_k \varphi}$$

$\Box$ rules

$$\frac{@_i [\mathrm{R}] \varphi \quad @_i \langle \mathrm{R} \rangle k}{@_k \varphi} \qquad \frac{\neg @_i [\mathrm{R}] \varphi}{\begin{array}{l} @_i \langle \mathrm{R} \rangle j \\ \neg @_j \varphi \end{array}}$$

Link with first-order deduction (Live Version)

Hybrid Logic	First Order Logic
$@_i \Diamond \phi$	

Link with first-order deduction (Live Version)

Hybrid Logic	First Order Logic
$@_i\Diamond\phi$	$\exists y(Riy \wedge ST_y(\phi))$

Link with first-order deduction (Live Version)

Hybrid Logic	First Order Logic
$@_i \Diamond \phi$	$\exists y(Riy \wedge ST_y(\phi))$
$@_i \Diamond j$ $@_j \phi$	

Link with first-order deduction (Live Version)

Hybrid Logic	First Order Logic
$@_i \Diamond \phi$	$\exists y(Riy \wedge ST_y(\phi))$
	$Rij \wedge ST_j(\phi)$
$@_i \Diamond j$	
$@_j \phi$	

Link with first-order deduction (Live Version)

Hybrid Logic	First Order Logic
$@_i\Diamond\phi$	$\exists y(Riy \wedge ST_y(\phi))$
	$Rij \wedge ST_j(\phi)$
$@_i\Diamond j$	Rij
$@_j\phi$	$ST_j(\phi)$

Link with first-order deduction (Studio Version)

- The hybrid rule from $@_i\Diamond\varphi$ conclude $@_i\Diamond j$ and $@_j\varphi$ is essentially the first-order rule of Existential Elimination (from $\exists x\varphi$ conclude $\varphi[x \leftarrow j]$).
- Recall that (via the Standard Translation) we know that $\Diamond\varphi$ is shorthand for $\exists y(Riy \wedge \mathrm{ST}_y(\varphi))$.
- Applying Existential Elimination to this yields $Rij \wedge \mathrm{ST}_j(\varphi)$. But this is just $@_i\Diamond j \wedge @_j\varphi$, the output of the tableau rule.
- In short, nominals give us exactly the grip we need on the bound variables hidden by modal notation. They give us the benefits of first-order techniques in a decidable logic.

Equality rules

But more rules are needed. Why? Nothing we have said so far gets to grips with fact that nominals have an intrinsic logic. Nominals give us a modal theory of equality, and we need to get to deal with this. Here's one way of doing this:

$$\frac{(i \text{ occurs on branch})}{@_i i} \qquad \frac{@_i j \quad @_i \varphi}{@_j \varphi} \qquad \frac{@_i \Diamond j \quad @_j k}{@_i \Diamond k}$$

$$(\Diamond p \wedge \Diamond \neg p) \to (\Box(q \to i) \to \Diamond \neg q)$$

$$(\Diamond p \wedge \Diamond \neg p) \rightarrow (\Box(q \rightarrow i) \rightarrow \Diamond \neg q)$$

1 $\neg @_i((\Diamond p \wedge \Diamond \neg p) \rightarrow (\Box(q \rightarrow i) \rightarrow \Diamond \neg q))$

$(\Diamond p \wedge \Diamond \neg p) \rightarrow (\Box(q \rightarrow i) \rightarrow \Diamond \neg q)$

1	$\neg @_i((\Diamond p \wedge \Diamond \neg p) \rightarrow (\Box(q \rightarrow i) \rightarrow \Diamond \neg q))$	
2	$@_i(\Diamond p \wedge \Diamond \neg p)$	
$2'$	$\neg @_i(\Box(q \rightarrow i) \rightarrow \Diamond \neg q)$	Propositional rule on 1

$(\Diamond p \wedge \Diamond \neg p) \to (\Box(q \to i) \to \Diamond \neg q)$

1	$\neg @_i((\Diamond p \wedge \Diamond \neg p) \to (\Box(q \to i) \to \Diamond \neg q))$	
2	$@_i(\Diamond p \wedge \Diamond \neg p)$	
2′	$\neg @_i(\Box(q \to i) \to \Diamond \neg q)$	Propositional rule on 1
3	$@_i \Diamond p$	
3′	$@_i \Diamond \neg p$	Propositional rule on 2

$(\Diamond p \wedge \Diamond \neg p) \to (\Box(q \to i) \to \Diamond \neg q)$

1	$\neg @_i((\Diamond p \wedge \Diamond \neg p) \to (\Box(q \to i) \to \Diamond \neg q))$	
2	$@_i(\Diamond p \wedge \Diamond \neg p)$	
2′	$\neg @_i(\Box(q \to i) \to \Diamond \neg q)$	Propositional rule on 1
3	$@_i \Diamond p$	
3′	$@_i \Diamond \neg p$	Propositional rule on 2
4	$@_i \Diamond j$	
4′	$@_j p$	$\Diamond$ rule on 3

$(\Diamond p \wedge \Diamond \neg p) \to (\Box(q \to i) \to \Diamond \neg q)$

1	$\neg @_i((\Diamond p \wedge \Diamond \neg p) \to (\Box(q \to i) \to \Diamond \neg q))$	
2	$@_i(\Diamond p \wedge \Diamond \neg p)$	
2′	$\neg @_i(\Box(q \to i) \to \Diamond \neg q)$	Propositional rule on 1
3	$@_i \Diamond p$	
3′	$@_i \Diamond \neg p$	Propositional rule on 2
4	$@_i \Diamond j$	
4′	$@_j p$	$\Diamond$ rule on 3
5	$@_i \Diamond k$	
5′	$@_k \neg p$	$\Diamond$ rule on 3’

$(\Diamond p \land \Diamond \neg p) \to (\Box(q \to i) \to \Diamond \neg q)$

1	$\neg @_i((\Diamond p \land \Diamond \neg p) \to (\Box(q \to i) \to \Diamond \neg q))$	
2	$@_i(\Diamond p \land \Diamond \neg p)$	
$2'$	$\neg @_i(\Box(q \to i) \to \Diamond \neg q)$	Propositional rule on 1
3	$@_i \Diamond p$	
$3'$	$@_i \Diamond \neg p$	Propositional rule on 2
4	$@_i \Diamond j$	
$4'$	$@_j p$	$\Diamond$ rule on 3
5	$@_i \Diamond k$	
$5'$	$@_k \neg p$	$\Diamond$ rule on 3'
6	$@_i \Box(q \to i)$	
$6'$	$\neg @_i \Diamond \neg q$	Propositional rule on 2'

The proof continued...

4	$@_i \Diamond j$
4′	$@_j \mathrm{p}$
5	$@_i \Diamond k$
5′	$@_k \neg \mathrm{p}$
6	$@_i \Box (\mathrm{q} \to i)$
6′	$\neg @_i \Diamond \neg \mathrm{q}$

The proof continued...

4	$@_i \Diamond j$	
4′	$@_j \mathrm{p}$	
5	$@_i \Diamond k$	
5′	$@_k \neg \mathrm{p}$	
6	$@_i \Box(\mathrm{q} \to i)$	
6′	$\neg @_i \Diamond \neg \mathrm{q}$	
7	$@_j \mathrm{q}$	$\neg\Diamond$ rule on 4 and 6', then $\neg @$ rule

The proof continued. . .

4	$@_i\Diamond j$	
4′	$@_j\mathrm{p}$	
5	$@_i\Diamond k$	
5′	$@_k\neg\mathrm{p}$	
6	$@_i\Box(\mathrm{q}\rightarrow i)$	
6′	$\neg@_i\Diamond\neg\mathrm{q}$	
7	$@_j\mathrm{q}$	$\neg\Diamond$ rule on 4 and 6', then $\neg@$ rule
8	$@_j(\mathrm{q}\rightarrow i)$	$\Box$ rule on 4 and 6

The proof continued. . .

4		$@_i\Diamond j$		
4′		$@_j \mathrm{p}$		
5		$@_i\Diamond k$		
5′		$@_k\neg \mathrm{p}$		
6		$@_i\Box(\mathrm{q} \to i)$		
6′		$\neg @_i\Diamond\neg \mathrm{q}$		
7		$@_j \mathrm{q}$		$\neg\Diamond$ rule on 4 and 6', then $\neg @$ rule
8		$@_j(\mathrm{q} \to i)$		$\Box$ rule on 4 and 6
9	$\neg @_j \mathrm{q}$		$@_j i$	Propositional rule on 7 and 8
	$\perp_{7,9}$			

The proof continued. . .

4	$@_i \Diamond j$		
4′	$@_j \mathrm{p}$		
5	$@_i \Diamond k$		
5′	$@_k \neg \mathrm{p}$		
6	$@_i \Box(\mathrm{q} \to i)$		
6′	$\neg @_i \Diamond \neg \mathrm{q}$		
7	$@_j \mathrm{q}$		$\neg\Diamond$ rule on 4 and 6', then $\neg @$ rule
8	$@_j(\mathrm{q} \to i)$		$\Box$ rule on 4 and 6
9	$\neg @_j \mathrm{q}$ $\perp_{7,9}$	$@_j i$	Propositional rule on 7 and 8

The proof continued...

4	$@_i \Diamond j$
4′	$@_j \mathrm{p}$
5	$@_i \Diamond k$
5′	$@_k \neg \mathrm{p}$
6	$@_i \Box(\mathrm{q} \to i)$
6′	$\neg @_i \Diamond \neg \mathrm{q}$
9	$@_j i$

The proof continued...

4	$@_i \Diamond j$	
4′	$@_j \mathrm{p}$	
5	$@_i \Diamond k$	
5′	$@_k \neg \mathrm{p}$	
6	$@_i \Box(\mathrm{q} \to i)$	
6′	$\neg @_i \Diamond \neg \mathrm{q}$	
9	$@_j i$	
10	$@_k \mathrm{q}$	$\neg\Diamond$ rule on 5 and 6', then $\neg @$ rule

The proof continued...

4	$@_i \Diamond j$	
4′	$@_j p$	
5	$@_i \Diamond k$	
5′	$@_k \neg p$	
6	$@_i \Box(q \to i)$	
6′	$\neg @_i \Diamond \neg q$	
9	$@_j i$	
10	$@_k q$	$\neg\Diamond$ rule on 5 and 6', then $\neg @$ rule
11	$@_k(q \to i)$	$\Box$ rule on 5 and 6

The proof continued...

4	$@_i \Diamond j$	
4′	$@_j \mathrm{p}$	
5	$@_i \Diamond k$	
5′	$@_k \neg \mathrm{p}$	
6	$@_i \Box(\mathrm{q} \to i)$	
6′	$\neg @_i \Diamond \neg \mathrm{q}$	
9	$@_j i$	
10	$@_k \mathrm{q}$	$\neg\Diamond$ rule on 5 and 6', then $\neg @$ rule
11	$@_k(\mathrm{q} \to i)$	$\Box$ rule on 5 and 6
12	$@_k i$	Modus Ponens on 10 and 11

The proof continued...

4	$@_i \Diamond j$	
4′	$@_j \mathrm{p}$	
5	$@_i \Diamond k$	
5′	$@_k \neg \mathrm{p}$	
6	$@_i \Box(\mathrm{q} \to i)$	
6′	$\neg @_i \Diamond \neg \mathrm{q}$	
9	$@_j i$	
10	$@_k \mathrm{q}$	$\neg\Diamond$ rule on 5 and 6', then $\neg @$ rule
11	$@_k(\mathrm{q} \to i)$	$\Box$ rule on 5 and 6
12	$@_k i$	Modus Ponens on 10 and 11
13	$@_i \mathrm{p}$	Nom on 4' and 9

The proof continued...

4	$@_i \Diamond j$	
4′	$@_j \mathrm{p}$	
5	$@_i \Diamond k$	
5′	$@_k \neg \mathrm{p}$	
6	$@_i \Box(\mathrm{q} \to i)$	
6′	$\neg @_i \Diamond \neg \mathrm{q}$	
9	$@_j i$	
10	$@_k \mathrm{q}$	$\neg\Diamond$ rule on 5 and 6', then $\neg @$ rule
11	$@_k(\mathrm{q} \to i)$	$\Box$ rule on 5 and 6
12	$@_k i$	Modus Ponens on 10 and 11
13	$@_i \mathrm{p}$	Nom on 4' and 9
14	$@_i \neg \mathrm{p}$	Nom on 5' and 12

The proof continued...

4	$@_i \Diamond j$	
4′	$@_j \mathrm{p}$	
5	$@_i \Diamond k$	
5′	$@_k \neg \mathrm{p}$	
6	$@_i \Box(\mathrm{q} \to i)$	
6′	$\neg @_i \Diamond \neg \mathrm{q}$	
9	$@_j i$	
10	$@_k \mathrm{q}$	$\neg\Diamond$ rule on 5 and 6', then $\neg @$ rule
11	$@_k(\mathrm{q} \to i)$	$\Box$ rule on 5 and 6
12	$@_k i$	Modus Ponens on 10 and 11
13	$@_i \mathrm{p}$	Nom on 4' and 9
14	$@_i \neg \mathrm{p}$	Nom on 5' and 12
15	$\neg @_i \mathrm{p}$	$\neg$ rule on 14 — contradiction!

Reasoning over other classes of models

- Our tableau system deals (correctly and completely) with reasoning over arbitrary models, that is, models where we have made no special assumptions about the underlying relations. For some applications this is sufficient.
- But (as we said at the start of the lecture) in many applications we are interested in models where the relations interpreting the modalities have special properties, such as symmetry, transitivity, irreflexivity, density, discreteness, antisymmetry, determinism, and so on. We need to find a way of coping with such frame conditions in hybrid logic.
- Our basic tableau system can easily be extended to cope with them, thus meeting the traditional modal goal of generality. We'll look at two examples.

Nice neighbours

Consider the following statement:

> *If you have a neighbour who only has nice neighbours, then you are nice.*

We can represent it as follows:

$$\langle \textsc{neighbour} \rangle \, [\textsc{neighbour}] \, \text{nice} \rightarrow \text{nice}$$

This is true no matter how the adjective "nice" is interpreted. Its truth hinges on the fact that neighbourhood is a symmetric relation.

Informal Argument

Informal Argument

- Suppose $\langle \textsc{neighbour} \rangle \, [\textsc{neighbour}] \, \mathrm{nice} \rightarrow \mathrm{nice}$ is false of you.

Informal Argument

- Suppose $\langle \textsc{neighbour} \rangle \, [\textsc{neighbour}] \, \text{nice} \to \text{nice}$ is false of you.
- Then $\langle \textsc{neighbour} \rangle \, [\textsc{neighbour}] \, \text{nice}$ is true of you, but nice is false of you (that is, you are not nice).

Informal Argument

- Suppose $\langle \textsc{neighbour} \rangle\, [\textsc{neighbour}]\, \mathrm{nice} \to \mathrm{nice}$ is false of you.
- Then $\langle \textsc{neighbour} \rangle\, [\textsc{neighbour}]\, \mathrm{nice}$ is true of you, but nice is false of you (that is, you are not nice).
- Then you have a neighbour (let's call him Julio) who only has nice neighbours (that is, $[\textsc{neighbour}]\, \mathrm{nice}$ is true of Julio).

Informal Argument

- Suppose $\langle \textsc{neighbour} \rangle \, [\textsc{neighbour}] \, \mathrm{nice} \to \mathrm{nice}$ is false of you.
- Then $\langle \textsc{neighbour} \rangle \, [\textsc{neighbour}] \, \mathrm{nice}$ is true of you, but nice is false of you (that is, you are not nice).
- Then you have a neighbour (let's call him Julio) who only has nice neighbours (that is, $[\textsc{neighbour}] \, \mathrm{nice}$ is true of Julio).
- But neighbourhood is a symmetric relation, hence you are one of Julio's neighbours.

Informal Argument

- Suppose $\langle \text{NEIGHBOUR} \rangle [\text{NEIGHBOUR}]\, \text{nice} \rightarrow \text{nice}$ is false of you.
- Then $\langle \text{NEIGHBOUR} \rangle [\text{NEIGHBOUR}]\, \text{nice}$ is true of you, but nice is false of you (that is, you are not nice).
- Then you have a neighbour (let's call him Julio) who only has nice neighbours (that is, $[\text{NEIGHBOUR}]\, \text{nice}$ is true of Julio).
- But neighbourhood is a symmetric relation, hence you are one of Julio's neighbours.
- But all Julio's neighbours are nice — so you must be nice too. Contradiction!

Informal Argument

- Suppose $\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \rightarrow \text{nice}$ is false of you.
- Then $\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice}$ is true of you, but nice is false of you (that is, you are not nice).
- Then you have a neighbour (let's call him Julio) who only has nice neighbours (that is, $[\text{NEIGHBOUR}]\, \text{nice}$ is true of Julio).
- But neighbourhood is a symmetric relation, hence you are one of Julio's neighbours.
- But all Julio's neighbours are nice — so you must be nice too. Contradiction!
- So $\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \rightarrow \text{nice}$ must true of you after all.

Informal Argument

- Suppose $\langle\text{NEIGHBOUR}\rangle\,[\text{NEIGHBOUR}]\,\text{nice} \to \text{nice}$ is false of you.
- Then $\langle\text{NEIGHBOUR}\rangle\,[\text{NEIGHBOUR}]\,\text{nice}$ is true of you, but nice is false of you (that is, you are not nice).
- Then you have a neighbour (let's call him Julio) who only has nice neighbours (that is, $[\text{NEIGHBOUR}]\,\text{nice}$ is true of Julio).
- But neighbourhood is a symmetric relation, hence you are one of Julio's neighbours.
- But all Julio's neighbours are nice — so you must be nice too. Contradiction!
- So $\langle\text{NEIGHBOUR}\rangle\,[\text{NEIGHBOUR}]\,\text{nice} \to \text{nice}$ must true of you after all.

But can we mimic this argument using our existing tableau system? Let's try. . .

$\langle$NEIGHBOUR$\rangle$ [NEIGHBOUR] nice $\rightarrow$ nice

$\langle \textsc{neighbour} \rangle\, [\textsc{neighbour}]\, \mathrm{nice} \rightarrow \mathrm{nice}$

1 $\neg @_i(\langle \textsc{neighbour} \rangle\, [\textsc{neighbour}]\, \mathrm{nice} \rightarrow \mathrm{nice})$

$\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \to \text{nice}$

1	$\neg @_i(\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \to \text{nice})$	
2	$@_i \langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice}$	
2′	$\neg @_i \text{nice}$	Propositional rule on 1

$\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \rightarrow \text{nice}$

1	$\neg @_i(\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \rightarrow \text{nice})$	
2	$@_i \langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice}$	
2′	$\neg @_i \text{nice}$	Propositional rule on 1
3	$@_i \langle \text{NEIGHBOUR} \rangle\, \mathit{julio}$	
3′	$@_{\mathit{julio}} [\text{NEIGHBOUR}]\, \text{nice}$	$\Diamond$ rule on 2

$\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \rightarrow \text{nice}$

1	$\neg @_i(\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \rightarrow \text{nice})$	
2	$@_i \langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice}$	
2′	$\neg @_i \text{nice}$	Propositional rule on 1
3	$@_i \langle \text{NEIGHBOUR} \rangle\, \mathit{julio}$	
3′	$@_{\mathit{julio}} [\text{NEIGHBOUR}]\, \text{nice}$	$\Diamond$ rule on 2

Now we are blocked. There is no way to close this branch.

But there is an easy solution

Add the following rule when working with symmetric relations:

$$\frac{@_i \langle \mathrm{NEIGHBOUR} \rangle\, j}{@_j \langle \mathrm{NEIGHBOUR} \rangle\, i}$$

(Here i and j are any nominals on the branch we are working on). This rule is a direct expression of symmetry, and with its help we can finish off our proof.

$\langle \mathrm{NEIGHBOUR} \rangle\, [\mathrm{NEIGHBOUR}]\, \mathrm{nice} \to \mathrm{nice}$

1	$i\,@_{julio}(\langle \mathrm{NEIGHBOUR} \rangle\, [\mathrm{NEIGHBOUR}]\, \mathrm{nice} \to \mathrm{nice})$	
2	$@_i \langle \mathrm{NEIGHBOUR} \rangle\, [\mathrm{NEIGHBOUR}]\, \mathrm{nice}$	
2′	$\neg @_i \mathrm{nice}$	Propositional rule on 1
3	$@_i \langle \mathrm{NEIGHBOUR} \rangle\, julio$	
3′	$@_{julio} [\mathrm{NEIGHBOUR}]\, \mathrm{nice}$	$\Diamond$ rule on 2

$\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \to \text{nice}$

1	$i\,@_{julio}(\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \to \text{nice})$	
2	$@_i \langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice}$	
2′	$\neg @_i \text{nice}$	Propositional rule on 1
3	$@_i \langle \text{NEIGHBOUR} \rangle\, julio$	
3′	$@_{julio} [\text{NEIGHBOUR}]\, \text{nice}$	$\Diamond$ rule on 2
4	$@_{julio} \langle \text{NEIGHBOUR} \rangle\, i$	Symmetry rule on 3

$\langle \text{NEIGHBOUR} \rangle \, [\text{NEIGHBOUR}] \, \text{nice} \rightarrow \text{nice}$

1	$i\,@_{julio}(\langle \text{NEIGHBOUR} \rangle \, [\text{NEIGHBOUR}] \, \text{nice} \rightarrow \text{nice})$	
2	$@_i \langle \text{NEIGHBOUR} \rangle \, [\text{NEIGHBOUR}] \, \text{nice}$	
$2'$	$\neg @_i \text{nice}$	Propositional rule on 1
3	$@_i \langle \text{NEIGHBOUR} \rangle \, julio$	
$3'$	$@_{julio} [\text{NEIGHBOUR}] \, \text{nice}$	$\Diamond$ rule on 2
4	$@_{julio} \langle \text{NEIGHBOUR} \rangle \, i$	Symmetry rule on 3
5	$@_i \text{nice}$	$\Box$ rule on 3' and 4

$\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \rightarrow \text{nice}$

1	$i\, @_{julio}(\langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice} \rightarrow \text{nice})$	
2	$@_i \langle \text{NEIGHBOUR} \rangle\, [\text{NEIGHBOUR}]\, \text{nice}$	
2′	$\neg @_i \text{nice}$	Propositional rule on 1
3	$@_i \langle \text{NEIGHBOUR} \rangle\, julio$	
3′	$@_{julio} [\text{NEIGHBOUR}]\, \text{nice}$	$\Diamond$ rule on 2
4	$@_{julio} \langle \text{NEIGHBOUR} \rangle\, i$	Symmetry rule on 3
5	$@_i \text{nice}$	$\Box$ rule on 3' and 4
	$\perp_{2',5}$	

Loop-free time

Consider the following statement:

> *If time i precedes time j, then time j does not precede time i.*

We can represent the statement as follows (where $\langle F \rangle$ is the Priorean diamond meaning "sometime-in-the-future"):

$$@_i \langle F \rangle\, j \to \neg @_j \langle F \rangle\, i$$

If you accept that temporal precedence is both transitive and irreflexive (the usual assumption) then this is a valid statement.

Informal Argument

Informal Argument

- Suppose that "if i precedes time j, then time j does not precede time i" is false.

Informal Argument

- Suppose that “if i precedes time j, then time j does not precede time i” is false.
- Then time i precedes time j, but time j precedes time i too.

Informal Argument

- Suppose that "if i precedes time j, then time j does not precede time i" is false.
- Then time i precedes time j, but time j precedes time i too.
- But temporal flow is transitive, so time i precedes time i.

Informal Argument

- Suppose that "if i precedes time j, then time j does not precede time i" is false.
- Then time i precedes time j, but time j precedes time i too.
- But temporal flow is transitive, so time i precedes time i.
- But temporal precedence is irreflexive, so time i cannot precede time i.

Informal Argument

- Suppose that "if i precedes time j, then time j does not precede time i" is false.
- Then time i precedes time j, but time j precedes time i too.
- But temporal flow is transitive, so time i precedes time i.
- But temporal precedence is irreflexive, so time i cannot precede time i.
- From this contradiction we conclude that our original statement was true after all.

But can we prove $@_i \langle \mathrm{F} \rangle j \rightarrow \neg @_j \langle \mathrm{F} \rangle i$ using our existing tableau system? Let's try...

But can we prove $@_i \langle \mathrm{F} \rangle j \rightarrow \neg @_j \langle \mathrm{F} \rangle i$ using our existing tableau system? Let's try...

1 $\neg @_k (@_i \langle \mathrm{F} \rangle j \rightarrow \neg @_j \langle \mathrm{F} \rangle i)$

But can we prove $@_i \langle F \rangle j \rightarrow \neg @_j \langle F \rangle i$ using our existing tableau system? Let's try...

1	$\neg @_k (@_i \langle F \rangle j \rightarrow \neg @_j \langle F \rangle i)$	
2	$@_k @_i \langle F \rangle j$	
$2'$	$\neg @_k \neg @_j \langle F \rangle i)$	Propositional rule on 1

But can we prove $@_i \langle F \rangle j \to \neg @_j \langle F \rangle i$ using our existing tableau system? Let's try...

1	$\neg @_k(@_i \langle F \rangle j \to \neg @_j \langle F \rangle i)$	
2	$@_k @_i \langle F \rangle j$	
2′	$\neg @_k \neg @_j \langle F \rangle i)$	Propositional rule on 1
3	$@_i \langle F \rangle j$	@ rule on 2

But can we prove $@_i\langle F\rangle j \rightarrow \neg @_j\langle F\rangle i$ using our existing tableau system? Let's try...

1	$\neg @_k(@_i\langle F\rangle j \rightarrow \neg @_j\langle F\rangle i)$	
2	$@_k @_i\langle F\rangle j$	
2′	$\neg @_k \neg @_j\langle F\rangle i)$	Propositional rule on 1
3	$@_i\langle F\rangle j$	$@$ rule on 2
4	$@_j\langle F\rangle i$	$\neg @ \neg$ rule on 2'

But can we prove $@_i \langle \mathrm{F} \rangle j \rightarrow \neg @_j \langle \mathrm{F} \rangle i$ using our existing tableau system? Let's try. . .

1	$\neg @_k(@_i \langle \mathrm{F} \rangle j \rightarrow \neg @_j \langle \mathrm{F} \rangle i)$	
2	$@_k @_i \langle \mathrm{F} \rangle j$	
2′	$\neg @_k \neg @_j \langle \mathrm{F} \rangle i)$	Propositional rule on 1
3	$@_i \langle \mathrm{F} \rangle j$	@ rule on 2
4	$@_j \langle \mathrm{F} \rangle i$	$\neg @ \neg$ rule on 2'

Now we are blocked. There is no way to close this branch.

But there is an easy solution

Add the following rules when working with irreflexive and transitive relations:

$$\frac{}{@_i \neg \langle \mathrm{F} \rangle\, i} \qquad\qquad \frac{@_i \langle \mathrm{F} \rangle\, j \qquad @_j \langle \mathrm{F} \rangle\, k}{@_i \langle \mathrm{F} \rangle\, k}$$

(Here i, j and k can be any nominals on the branch we are working on).

These rules are a direct expression of irreflexivity and transitivity, and with their help we can finish off our proof.

$@_i \langle F \rangle j \rightarrow \neg @_j \langle F \rangle i$

1	$\neg @_k (@_i \langle F \rangle j \rightarrow \neg @_j \langle F \rangle i)$	
2	$@_k @_i \langle F \rangle j$	
$2'$	$\neg @_k \neg @_j \langle F \rangle i)$	Propositional rule on 1
3	$@_i \langle F \rangle j$	$@$ rule on 2
4	$@_j \langle F \rangle i$	$\neg @ \neg$ rule on 2'

$@_i \langle F \rangle j \to \neg @_j \langle F \rangle i$

1	$\neg @_k(@_i \langle F \rangle j \to \neg @_j \langle F \rangle i)$	
2	$@_k @_i \langle F \rangle j$	
$2'$	$\neg @_k \neg @_j \langle F \rangle i)$	Propositional rule on 1
3	$@_i \langle F \rangle j$	@ rule on 2
4	$@_j \langle F \rangle i$	$\neg @ \neg$ rule on 2'
5	$@_i \langle F \rangle i$	Transitivity rule on 3 and 4

$@_i \langle F \rangle j \rightarrow \neg @_j \langle F \rangle i$

1	$\neg @_k (@_i \langle F \rangle j \rightarrow \neg @_j \langle F \rangle i)$	
2	$@_k @_i \langle F \rangle j$	
$2'$	$\neg @_k \neg @_j \langle F \rangle i)$	Propositional rule on 1
3	$@_i \langle F \rangle j$	$@$ rule on 2
4	$@_j \langle F \rangle i$	$\neg @ \neg$ rule on 2'
5	$@_i \langle F \rangle i$	Transitivity rule on 3 and 4
6	$\neg @_i \langle F \rangle i$	Irreflexivity rule

$@_i\langle \mathrm{F}\rangle\, j \to \neg @_j\langle \mathrm{F}\rangle\, i$

1	$\neg @_k(@_i\langle \mathrm{F}\rangle\, j \to \neg @_j\langle \mathrm{F}\rangle\, i)$	
2	$@_k @_i\langle \mathrm{F}\rangle\, j$	
$2'$	$\neg @_k \neg @_j\langle \mathrm{F}\rangle\, i)$	Propositional rule on 1
3	$@_i\langle \mathrm{F}\rangle\, j$	@ rule on 2
4	$@_j\langle \mathrm{F}\rangle\, i$	¬@¬ rule on 2'
5	$@_i\langle \mathrm{F}\rangle\, i$	Transitivity rule on 3 and 4
6	$\neg @_i\langle \mathrm{F}\rangle\, i$	Irreflexivity rule
	$\perp_{5,6}$	

Pure formulas

- It's time to be more precise about what completeness results are possible here.
- To do this we need to think about pure formulas.
- A formula of the basic hybrid language is pure if it contains no propositional variables. That is, the only atoms in pure formulas are nominals (and $\top$ and $\bot$ if we have them in the language).
- We'll first discuss what we can say about frames using pure formulas, and then we'll state a general result about how they can help us in hybrid deduction.

Frame definability (I)

A formula defines a class of frames if it is valid on precisely the frames belonging to that class class. We can define many important classes of frames using pure formulas:

$@_i \Diamond i$	*Reflexivity*
$@_i \Diamond j \rightarrow @_j \Diamond i$	*Symmetry*
$@_i \Diamond j \wedge @_j \Diamond k \rightarrow @_i \Diamond k$	*Transitivity*

Frame definability (II)

These previous three examples are also definable using orthodox modal language. But pure formulas can also define frame classes which are not definable in orthodox modal logic:

$@_i \neg \Diamond i$	*Irreflexivity*
$@_i \Diamond j \to @_j \neg \Diamond i$	*Asymmetry*
$@_i \Box (\Diamond i \to i)$	*Antisymmetry*
$@_j \Diamond i \vee @_j i \vee @_i \Diamond j$	*Trichotomy*

From formulas to tableau rules

Let $@_i\varphi$ be a pure formula, built out of nominals $i, i_1, \ldots, i_n$. Then the simplest (though not always the smartest!) way of turning this formula into a tableau rule is as follows:

$$\frac{(j, j_1, \ldots, j_n \text{ on branch})}{@_i\varphi[i \leftarrow j, i_1 \leftarrow j_1, \ldots, i_n \leftarrow j_n]}$$

This rule simply says: for any branch B of the tableau you are building, you are free to instantiate $@_i\varphi$ with nominals occurring on B and add the resulting formula to the end of B.

Frame definability and deduction match for pure formulas

Completeness Theorem Suppose you extend the basic tableau system with the tableau rules for the pure formulas $@_j\varphi, \ldots, @_k\psi$ (that is, the rules of the form just described). Then the resulting system is (sound and) complete with respect to the class of frames defined by these formulas.

That is, the frame-defining and deductive powers of pure formulas match perfectly for pure formulas.

Two comments should be made about this result. . .

We can use any pure formula

At first glance, it seems that this completeness result only covers pure formulas of the form $@_i\varphi$. But many interesting pure formulas are not of this form. For example symmetry: $@_i\Diamond j \rightarrow @_j\Diamond i$.

Note, however, that for any pure formula φ, and any nominals i, φ and $@_i\varphi$ define exactly the same class of frames.

For example symmetry can be defined by $@_k(@_i\Diamond j \rightarrow @_j\Diamond i)$.

So our completeness theorem is fully general: it covers all classes of frames definable by a pure formulas.

But we can often be smarter

Suppose we want a complete system for symmetry. We could do this by adding the rule suggested by the previous system:

$$\frac{}{@_k(@_i\Diamond j \rightarrow @_j\Diamond i)}.$$

But in the nice neighbours example we used the following rule instead:

$$\frac{@_i\Diamond j}{@_j\Diamond i}$$

This rule is smarter: it saves us having to use tableau rules to get rid of the outermost $@_k$, and then break down the implication.

Slightly more generally

Given a pure formula of the form

$$(@_i\varphi_1 \wedge \cdots \wedge @_j\varphi_n) \to (@_k\varphi_{n+1} \vee \cdots \vee @_l\varphi_{n+m})$$

we can turn it into the tableau rule

$$\frac{@_i\varphi_1, \ldots, @_j\varphi_n}{@_k\varphi_{n+1} \mid \cdots \mid @_l\varphi_{n+m}}$$

without losing completeness.

Further themes in hybrid deduction

To conclude, let's briefly address the following questions:

- Why are general completeness proof easy to come by in hybrid logic?
- Can we really adapt these ideas to other proof styles?
- Is any of this stuff implementable?

Why are general completeness proofs so easy to come by in hybrid logic?

- Essentially because the basic hybrid logic enables us to use first-order techniques to build models.
- For example, when proving completeness for hybrid Hilbert systems, it's not necessary to use modal-style canonical models — you can build what are basically first-order Henkin models.
- And for tableau completeness proofs, observe that the tableau rules crunch formulas down into expressions of the form $(\neg)@_i\mathrm{p}$, $(\neg)@_i j$ and $(\neg)@_i\Diamond j$. Open branches are thus Robinson diagrams of satisfying models.

Named models are important

- Moreover, the models we build in this way are named. (A named model is a model in which every point is named by some nominal.)
- A simple model theoretic argument shows that if all instances of a pure formula φ are true at all states in a named model, then the underlying frame validates ϕ. This gives us completeness for any extension by pure axioms.

Can we really adapt these ideas to other proof styles?

Yes. The key insight is that the combination of nominals and @ allow us to extract information from behind the scope of diamonds.

This idea has been successfully applied to define general sequent calculi (Seligman), natural deduction systems (Seligman, Braüner), and resolution calculi (Areces), and display calculi (Demri and Goré).

Let's take a quick look at the way Torben Braüner handles natural deduction in hybrid logic.

Some basic natural deduction rules

$$\dfrac{\begin{matrix}[@_i\varphi] \\ \vdots \\ @_i\psi\end{matrix}}{@_i(\varphi \to \psi)}\ (\to I) \qquad \dfrac{@_i(\varphi \to \psi) \qquad @_i\varphi}{@_i\psi}\ (\to E)$$

$$\dfrac{@_i\varphi}{@_k@_i\varphi}\ (@I) \qquad \dfrac{@_k@_i\varphi}{@_i\varphi}\ (@E)$$

Natural deduction rules for modalities

$$\dfrac{\begin{array}{c}[@_i\Diamond j]\\ \vdots\\ @_j\varphi\end{array}}{@_i\Box\varphi}\ (\Box I)^*
\qquad\qquad
\dfrac{@_i\Box\varphi \quad @_i\Diamond k}{@_k\varphi}\ (\Box E)$$

$*$ j does not occur in $@_i\Box\varphi$ or in any undischarged assumptions other than the specified occurrences of $@_i\Diamond j$.

An example: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[@_i\Box(\varphi \to \psi)]^3 \quad [@_i\Diamond j]^1}{@_j(\varphi \to \psi)}\,(\Box E) \qquad \dfrac{[@_i\Box\varphi]^2 \quad [@_i\Diamond j]^1}{@_j\varphi}\,(\Box E)}{@_j\psi}\,(\to E)}{@_i\Box\psi}\,(\Box I)^1}{@_i(\Box\varphi \to \Box\psi)}\,(\to I)^2}{@_i(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi))}\,(\to I)^3$$

Is any of this stuff implementable?

Yes — but we need to be careful. For example, the equality rules discussed today are nice for hand calculation, but naive computationally.

The **HTab** system (Areces and Hoffmann) implements more sophisticated rules (due to Bolander and Blackburn) which guarantee termination. The system is optimised in several ways, and is a competitive prover.

Though not as fast as the more recent **Spartacus** prover (Smolka and Kaminski).

And then there's resolution

A significant development is the adaptation of the resolution method for hybrid logic (Areces) and the implementation of the **HyLoRes** prover (Areces, Gorín, and Heguiabehere).

Strictly speaking, the method is resolution, plus a little paramodulation to handle the equality reasoning. The hybrid resolution rule is significantly simpler than other known approaches to modal resolution — @ and nominals allow us to pull resolvents out of the scope of modalities.

Many first-order resolution optimization techniques transfer to hybrid logic, and Areces and Gorin have incorporated such improvements into **HyLoRes**.

Summing up . . .

- Orthodox modal logic demands proof methods that are applicable to a wide range to of logics. But because it is hard to extract information from under the scope of diamonds it has been forced to rely on Hilbert-systems, thereby sacrificing ease-of-use.
- The new tools offered by the basic hybrid language (nominals and @) enable us to define usable proof systems, such as tableau and natural deduction, basically because they make it easy to pull information out of modal scope.
- These proof methods can be generalized to a wide range of logics (completeness is automatic for pure formulas). Mature implementations now exist.