Model theory (analytic part)

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Days in Logic 2014

The tutorial

The tutorial

- A bit of o-minimality

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- A bit of o-minimality
- A bit of o-minimality and Gronthendieck

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- A bit of o-minimality
- A bit of o-minimality and Gronthendieck
- A bit of o-minimality and André-Oort.

A bit of o-minimality

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Tame vs wild topology

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Lets consider the question tame vs wild topology in the following setting:

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Consider a mathematical structure

$$\mathcal{M} = (M, (c)_{c \in \mathcal{C}}, (f)_{f \in \mathcal{F}}, (R)_{R \in \mathcal{R}}, <)$$

where $(M, <)$ is a dense total order with no endpoints.

Tame vs wild topology

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Consider the category

$$\mathrm{Def}(\mathcal{M})$$

of definable sets and maps in $\mathcal{M}$.

Tame vs wild topology

Consider the category

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of <u>definable sets and maps</u> in $\mathcal{M}$.

- X is object of $\mathrm{Def}(\mathcal{M})$ iff $X \subseteq M^n$ (some n) and is defined by a first-order formula (with parameters) in the language of $\mathcal{M}$.
- $f : X \to Y$ is morphism of $\mathrm{Def}(\mathcal{M})$ iff its graph $\Gamma(f)$ is an object of $\mathrm{Def}(\mathcal{M})$.

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What can be said about the topology of the objects of

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- Poincaré paradise: with some glimpses of the rich algebraic-analytic-topological structure of the continuum.

Tame vs wild topology

What can be said about the topology of the objects of

$$\mathrm{Def}(\mathcal{M})?$$

- Poincaré paradise: with some glimpses of the rich algebraic-analytic-topological structure of the continuum.
- Cantor paradise: where only set theoretic features ultimately survive (Cantor/Borel like sets).

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To know in which paradise we enter it is useful to know:

Tame vs wild topology

To know in which paradise we enter it is useful to know:

Defining formula	Defined set
$\phi(\overline{x})$	A
$\psi(\overline{x})$	B
$\neg\phi(\overline{x})$	$M^n \setminus A$
$\phi(\overline{x}) \wedge \psi(\overline{x})$	$A \cap B$
$\phi(\overline{x}) \vee \psi(\overline{x})$	$A \cup B$
$\exists x_i \phi(\overline{x})$	$\pi_i^n(A)$

where $\pi_i^n(x_1, \ldots, x_n) = (x_1, \ldots, \widehat{x_i}, \ldots, x_n)$.

Tame vs wild topology

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where $\pi_i^n(x_1, \ldots, x_n) = (x_1, \ldots, \widehat{x_i}, \ldots, x_n)$. If we know QF

definable sets we must <u>know the effect</u> of

$$\setminus \quad \cap \quad \cup \quad \pi_i^n$$

Tame vs wild topology

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What should one avoid to remain tame?

Tame vs wild topology

What should one avoid to remain tame?

- (Kechris book) Let

  $$\mathcal{M} = (\mathbb{R}, (c)_{c \in \mathcal{C}}, (f)_{f \in \mathcal{F}}, (R)_{R \in \mathcal{R}}, <)$$

  and suppose that $(\mathbb{R}, 0, 1, +, \cdot, <)$ and $\mathbb{Z}$ are in $\mathrm{Def}(\mathcal{M})$. Then X in $\mathrm{Def}(\mathcal{M})$ iff X is a projective subset of some $\mathbb{R}^n$. In particular, all Borel subsets of some $\mathbb{R}^n$ are in $\mathrm{Def}(\mathcal{M})$.

Tame vs wild topology

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- So one should avoid having $(\mathbb{Z}, 0, 1, +, \cdot)$ in $\mathrm{Def}(\mathcal{M})$.

Tame vs wild topology

Tame vs wild topology

What exactly is to be tame? Well...it is too vague, but in

Grothendieck's *Esquisse d'un programme* (unpublished research proposal from 1984)

proposes an axiomatic development of topology based on stratifications that should include as special cases:

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- Real algebraic and semi-algebraic geometry.

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proposes an axiomatic development of topology based on <u>stratifications</u> that should <u>include</u> as special cases:

- Real algebraic and semi-algebraic geometry.
- Semi-analytic and sub-analytic geometry.

O-minimal structures

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O-minimal structures are a class of ordered structures which are a model theoretic (logic) generalization of interesting classical structures such as:

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- the field of real numbers

$$\overline{\mathbb{R}} = (\mathbb{R}, 0, 1, -, +, \cdot, <);$$

- the field of real numbers expanded by restricted analytic functions

$$\overline{\mathbb{R}}_{\mathrm{an}} = (\mathbb{R}, 0, 1, -, +, \cdot, (f)_{f \in \mathrm{an}}, <).$$

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Hopefully o-minimal structures:

- capture tameness;
- provide new insights originated from model-theoretic methods into the real analytic-like setting.

O-minimal structures

O-minimal structures

So what is an o-minimal structure?

O-minimal structures

So what is an o-minimal structure?

(van den Dries 84; Pillay and Steinhorn 86):

$$\mathcal{M} = (M, (c)_{c \in \mathcal{C}}, (f)_{f \in \mathcal{F}}, (R)_{R \in \mathcal{R}}, <)$$

is o-minimal if every definable subset of M in the structure $\mathcal{M}$ is already definable in the dense linearly ordered set

$$(M, <).$$

The simplest example

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Let's find out what are the subset of M definable in the dense linearly ordered set

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QF definable subsets of M^n are finite unions of simple sets:

$$\{\overline{x} \in M^n : \begin{cases} f_{i_1}(\overline{x}) = f_{i'_1}(\overline{x}) \\ \vdots \\ f_{i_k}(\overline{x}) = f_{i'_k}(\overline{x}) \end{cases} \quad \text{and} \quad \begin{cases} f_{j_1}(\overline{x}) < f_{j'_1}(\overline{x}) \\ \vdots \\ f_{j_s}(\overline{x}) < f_{j'_s}(\overline{x}) \end{cases} \}$$

where

$$\{i_1, \dots i_k\}, \{i'_1, \dots, i'_k\}, \{j_1, \dots, j_s\}, \{j'_1, \dots, j_s\} \subseteq \{1, \dots, N\}$$

and each f_l ($l = 1, \dots N$) is a simple function, i.e.:

$$f_l = x_l \quad \text{or} \quad f_l = c_l \in M.$$

The simplest example

The simplest example

The collection of finite unions of simple sets is closed under the operations

$$\setminus \quad \cap \quad \cup$$

What about under the operations

$$\pi_i^n?$$

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Yes by...

The simplest example

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Theorem

The first-order theory of DLO has QE.

The simplest example

Theorem

The first-order theory of DLO has QE.

Corollary

$A \subseteq M^n$ definable in $(M, <)$ iff X is finite unions of simple sets.

Corollary

$A \subseteq M$ be definable in $(M, <)$ iff A is a finite union of sets of the form $\{a\}$, $(-\infty, a), (a, b)$ or $(b, +\infty)$ with $a, b \in M$.

The simplest example

The simplest example

A geometric approach is instructive.....

The simplest example

A geometric approach is instructive.....

Proposition (Cell decomposition)

Let $f_1, \ldots f_N : M^{n+1} \to M$ be simple. Then exists a partition $\{C_1, \ldots, C_k\}$ of M^n by simple sets and functions $\zeta_{C,j} : C \to M$ ($C \in \{C_1, \ldots, C_k\}$ and $j = 1, \ldots, j(C)$) such that:

(i) $\zeta_{C,1} < \ldots < \zeta_{C,j(C)}$

(ii) $\zeta_{C,j}$ is the restriction of a simple function.

(iii) for each $(p, q) \in N^2$ we have either $f_p < f_q$ or $f_p = f_q$ or $f_q < f_p$ on the simple sets

$$\Gamma(\zeta_{C,j}),\ (-\infty, \zeta_{C,1}),\ (\zeta_{C,j}, \zeta_{C,j+1}) \text{ and } (\zeta_{C,j(C)}, +\infty).$$

The simplest example

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Proof. By induction on n. Let $s \leq N$ be such that for each $l \leq s$, $f_l : M^{n+1} \to M$ is not x_i and consider $\overline{f}_l : M^n \to M$ simple given by f_l. Apply Proposition to $\overline{f}_1, \ldots, \overline{f}_s$ and take $\{C_1, \ldots, C_k\}$ a partition of M^n by simple sets given by (iii).

For $C \in \{C_1, \ldots, C_k\}$ let

$$\zeta_{C,j} = \overline{f}_{s+j|C} : C \to M$$

with $j = 1, \ldots, j(C) = N - s$.

The simplest example

The simplest example

Let B be one of the

$$\Gamma(\zeta_{C,j}),\ (-\infty,\zeta_{C,1}),\ (\zeta_{C,j},\zeta_{C,j+1}) \text{ and } (\zeta_{C,j(C)},+\infty).$$

Let $\epsilon_{(l,r)} = \mathrm{sign}(f_{l|B}, f_{r|B})$ (obvious definition) and let

$$B' = \{(x,t) \in C \times V : \mathrm{sign}(f_{l|}(x,t), f_{r|}(x,t)) = \epsilon_{(l,r)}, \forall (l,r)\}$$

Then $B = B'$.
If $B' \setminus B \neq \emptyset$, fix $(x,t') \in B'$ and take $(x,t) \in B$ say $t < t'$.Since $\{s \in V : (x,s) \in B'\}$ is an interval (easy), $[t,t'] \subseteq B'$. Contradiction: one $(f_{l|B'}, f_{r|B'})$ must change sign and it does not change sign. $\square$

The simplest example

The simplest example

Corollary

Let $A \subseteq M^{n+1}$ be a finite union of simple sets. Then A is a finite union of simple sets of the form

$$\Gamma(\zeta_{C,j}),\ (-\infty, \zeta_{C,1}),\ (\zeta_{C,j}, \zeta_{C,j+1}) \text{ and } (\zeta_{C,j(C)}, +\infty).$$

In particular, $\pi_i^{n+1}(A)$ is a finite union of simple sets.

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In particular, $\pi_i^{n+1}(A)$ is a finite union of simple sets.

Corollary

Let $A \subseteq M$ be definable in $(M, <)$. Then A is a finite union of sets of the form $\{a\}$, $(-\infty, a), (a, b)$ or $(b, +\infty)$ with $a, b \in M$.

PL geometry

PL geometry

Let's find out what are the subset of V definable in

$$(V, 0, -, +, (\lambda)_{\lambda \in D}, <)$$

an ordered vector space over an ordered division ring D with $(V, 0, -, +)$ an ordered divisible abelian group.

PL geometry

Let's find out what are the subset of V definable in

$$(V, 0, -, +, (\lambda)_{\lambda \in D}, <)$$

an ordered vector space over an ordered division ring D with $(V, 0, -, +)$ an ordered divisible abelian group.

QF definable subsets of V^n are finite unions of <u>basic semi-linear sets</u>:

$$\{\overline{x} \in V^n : \begin{cases} f_1(\overline{x}) = 0 \\ \vdots \\ f_t(\overline{x}) = 0 \end{cases} \quad \text{and} \quad \begin{cases} g_1(\overline{x}) > 0 \\ \vdots \\ g_s(\overline{x}) > 0 \end{cases} \}$$

where each f_l and each g_k is an <u>affine function</u>, i.e.:

$$h(\overline{x}) = \lambda_1 x_1 + \ldots + \lambda_n x_n + c.$$

PL geometry

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The collection of semi-linear sets i.e., finite unions of simple sets, is closed under the operations

$$\setminus \quad \cap \quad \cup$$

What about under the operations

$$\pi_i^n?$$

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PL geometry is the geometry of semi-linear spaces.

PL geometry

Theorem

The first-order theory of ordered vector spaces over ordered division rings which are ordered divisible abelian groups has QE.

PL geometry

Theorem
The first-order theory of ordered vector spaces over ordered division rings which are ordered divisible abelian groups has QE.

Corollary
$A \subseteq V^n$ definable in $(V, 0, -, +, (\lambda)_{\lambda \in D}, <)$ iff A is a semi-linear set.

Corollary
$(V, 0, -, +, (\lambda)_{\lambda \in D}, <)$ is o-minimal.

PL geometry

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Proposition (Cell decomposition)

Let $f_1, \ldots f_N : V^{n+1} \to V$ be affine. Then exists a partition $\{C_1, \ldots, C_k\}$ of V^n by basic semi-linear sets and functions $\zeta_{C,j} : C \to V$ ($C \in \{C_1, \ldots, C_k\}$ and $j = 1, \ldots, j(C)$) such that:

(i) $\zeta_{C,1} < \ldots < \zeta_{C,j(C)}$

(ii) $\zeta_{C,j}$ is the restriction of an affine function.

(iii) each f_l has constant sign on the basic semi-linear sets

$$\Gamma(\zeta_{C,j}),\ (-\infty, \zeta_{C,1}),\ (\zeta_{C,j}, \zeta_{C,j+1}) \text{ and } (\zeta_{C,j(C)}, +\infty).$$

PL geometry

Proof. By induction on n. Let $s \leq N$ be such that for each $l \leq s$, $f_l : V^{n+1} \to V$ is does not depend on x_i and consider $\overline{f}_l : V^n \to V$ affine given by f_l. For $s < l \leq N$ let $\overline{f}_l : V^n \to V$ be

$$\overline{f}_l(\overline{x}) = (\lambda_i^l)^{-1}(-f_l(\overline{x}) - \lambda_i x_i)$$

which for each fixed $(x_1, \ldots, \widehat{x_i}, \ldots, x_n)$ gives the only zero of f_l, where

$$f_l(\overline{x}) = \lambda_1^l x_1 + \ldots + \lambda_i^l x_i + \ldots + \lambda_{n+1}^l x_{n+1} + c.$$

Apply Proposition to $\{\overline{f}_1, \ldots, \overline{f}_s\} \cup \{\overline{f}_p - \overline{f}_q : s < p, q \leq N\}$ and take $\{C_1, \ldots, C_k\}$ a partition of V^n by basic semi-linear sets given by (iii). For $C \in \{C_1, \ldots, C_k\}$ let $\zeta_{C,j} = \overline{f}_{s+j|C} : C \to V$ with $j = 1, \ldots, j(C) = N - s$.

PL geometry

Let B be one of the

$$\Gamma(\zeta_{C,j}),\ (-\infty,\zeta_{C,1}),\ (\zeta_{C,j},\zeta_{C,j+1}) \text{ and } (\zeta_{C,j(C)},+\infty).$$

Let $\epsilon_l = \mathrm{sign}(f_{l|B})$ and let

$$B' = \{(x,t) \in C \times V : \mathrm{sign}(f_{l|}(x,t)) = \epsilon_l,\ \forall l\}$$

<u>Then $B = B'$.</u>
If $B' \setminus B \neq \emptyset$, fix $(x,t') \in B'$ and take $(x,t) \in B$ say $t < t'$.Since $\{s \in V : (x,s) \in B'\}$ is an interval (<u>easy</u>), $[t,t'] \subseteq B'$.
Contradiction: one $f_{l|B'}$ must change sign and it does not change sign. $\square$

PL geometry

Corollary

Let $A \subseteq V^{n+1}$ be a semi-linear set. Then A is a finite union of basic semi-linear sets of the form

$$\Gamma(\zeta_{C,j}),\ (-\infty, \zeta_{C,1}),\ (\zeta_{C,j}, \zeta_{C,j+1}) \text{ and } (\zeta_{C,j(C)}, +\infty).$$

In particular, $\pi_i^{n+1}(A)$ is a semi-linear set.

PL geometry

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$$\Gamma(\zeta_{C,j}),\ (-\infty, \zeta_{C,1}),\ (\zeta_{C,j}, \zeta_{C,j+1}) \text{ and } (\zeta_{C,j(C)}, +\infty).$$

In particular, $\pi_i^{n+1}(A)$ is a semi-linear set.

Corollary

Let $A \subseteq V$ be definable in $(V, 0, -, +, (\lambda)_{\lambda \in D}, <)$. Then A is a finite union of sets of the form $\{a\}$, $(-\infty, a), (a, b)$ or $(b, +\infty)$ with $a, b \in V$.

Semi-algebraic geometry

Semi-algebraic geometry

Let's find out what are the subset of R definable in

$$(R, 0, 1, -, +, \cdot, <)$$

a real closed ordered field.

Semi-algebraic geometry

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QF definable subsets of R^n are <u>semi-algebraic sets</u> i.e., finite unions of sets of the form:

$$\{\overline{x} \in R^n : \begin{cases} f_1(\overline{x}) = 0 \\ \vdots \\ f_t(\overline{x}) = 0 \end{cases} \quad \text{and} \quad \begin{cases} g_1(\overline{x}) > 0 \\ \vdots \\ g_s(\overline{x}) > 0 \end{cases} \}$$

where each f_l and each g_k is a <u>polynomial</u> in $R[X_1, \ldots, X_n]$.

Semi-algebraic geometry

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The collection of semi-algebraic sets is closed under the operations

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Semi-algebraic geometry is the geometry of semi-algebraic (subsets of real algebraic) spaces.

Semi-algebraic geometry

Semi-algebraic geometry

Theorem (Tarski-Seidenberg)

The first-order theory of RCF has QE.

Semi-algebraic geometry

Theorem (Tarski-Seidenberg)

The first-order theory of RCF has QE.

Corollary

$A \subseteq R^n$ definable in $(R, 0, 1, -, +, \cdot, <)$ iff A is a semi-algebraic set.

Corollary

$(R, 0, 1, -, +, \cdot, <)$ is o-minimal.

Semi-algebraic geometry

Semi-algebraic geometry

A geometric approach is instructive.....

Semi-algebraic geometry

A geometric approach is instructive.....

Theorem (Cell decomposition)

Let $f_1, \ldots f_N$ be in $R[X_1, \ldots, X_n][T]$ such that all non zero $\frac{\partial^r f_l}{\partial T^r}$ are in list. Then exists a partition $\{C_1, \ldots, C_k\}$ of R^n by semi-algebraic sets and functions $\zeta_{C,j} : C \to R$ ($C \in \{C_1, \ldots, C_k\}$ and $j = 1, \ldots, j(C)$) such that:

(i) $\zeta_{C,1} < \ldots < \zeta_{C,j(C)}$

(ii) $\zeta_{C,j}$ is continuous and semi-algebraic (i.e., the graph is semi-algebraic).

(iii) each f_l has constant sign on the semi-algebraic sets

$$\Gamma(\zeta_{C,j}),\ (-\infty, \zeta_{C,1}),\ (\zeta_{C,j}, \zeta_{C,j+1}) \text{ and } (\zeta_{C,j(C)}, +\infty).$$

Semi-algebraic geometry

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The proof is bit harder...

Semi-algebraic geometry

The proof is bit harder...

due to Łojasiewicz (1965) in the real field

$$\overline{\mathbb{R}} = (\mathbb{R}, 0, 1, -, +, \cdot, <);$$

other proofs by Bochnak and Efroymson (1980), Delzell (1982), Bochnak, Coste and Roy (1987), van den Dries (1982).

Semi-algebraic geometry

Semi-algebraic geometry

Sketch of the proof... first we need

Semi-algebraic geometry

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Lemma (Thom's lemma)

Let $f_1, \ldots f_N$ be in $\mathbb{R}[T]$ such that all non zero $\frac{df_l}{dT}$ are in the list. For each l, let $\epsilon_l \in \{-1, 0, 1\}$. Then

$$\{x \in \mathbb{R} : \mathrm{sign}(f_l(x)) = \epsilon_l,\ l = 1, \ldots, N\}$$

is either empty, a point or an open interval. Moreover, the closure of such set is obtained by relaxing the sign conditions.

Semi-algebraic geometry

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Lemma (Thom's lemma)

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is either empty, a point or an open interval. Moreover, the closure of such set is obtained by relaxing the sign conditions.

Proof. By induction on N assuming $\deg(f_i) \leq \deg(f_{i+1})$. Case $N = 0$ is obvious and when we add f_N note that $\frac{df_N}{dT} \in \{f_1, \dots f_{N-1}\}$ and has constant sign on

$$\{x \in \mathbb{R} : \mathrm{sign}(f_l(x)) = \epsilon_l,\ l = 1, \dots, N-1\}.$$

□

Semi-algebraic geometry

Semi-algebraic geometry

Sketch of the proof of Cell decomposition...

Semi-algebraic geometry

Sketch of the proof of Cell decomposition...

By induction on n. Take

$$f_\Delta(T) = \Pi_{(l,r)\in\Delta} \frac{\partial^r f_l}{\partial T^r}(T),$$

$$e = 0, 1, \ldots, \max\{\deg(f_\Delta(T))\}$$

$$A_{\Delta,e} = \{x \in \mathbb{R}^n : \#\{z \in \mathbb{C} : f_\Delta(x, z) = 0\} = e\}$$

<u>Claim 1:</u> each $A_{\Delta,e}$ is semi-algebraic subset of $\mathbb{R}^n$.

Semi-algebraic geometry

Sketch of the proof of Cell decomposition...

By induction on n. Take

$$f_\Delta(T) = \Pi_{(I,r)\in\Delta} \frac{\partial^r f_I}{\partial T^r}(T),$$

$$e = 0, 1, \ldots, \max\{\deg(f_\Delta(T))\}$$

$$A_{\Delta,e} = \{x \in \mathbb{R}^n : \#\{z \in \mathbb{C} : f_\Delta(x,z) = 0\} = e\}$$

Claim 1: each $A_{\Delta,e}$ is semi-algebraic subset of $\mathbb{R}^n$.

(A special and easy case of Chevalley's constructibility theorem in $(\mathbb{C}, 0, 1, +, -, \cdot)$, i.e., QE for ACF...)

Semi-algebraic geometry

Semi-algebraic geometry

Apply the Theorem to all the $A_{\Delta,e}$ and take $\{C_1, \ldots, C_k\}$ partition of $\mathbb{R}^n$ by semi-algebraic sets given by (iii). Let $C \in \{C_1, \ldots, C_k\}$. Then exists $\mu(C)$ such that $C \subseteq A_{\Delta,\mu(C)}$.
Claim 2: exists $j(C)$ such that

$$\#\{y \in \mathbb{R} : f_\Delta(x, y) = 0\} = j(C) \text{ for all } x \in C,$$

moreover, the $\zeta_{C,j} : C \to \mathbb{R}$ ($j = 1, \ldots, j(C)$) giving the roots are continuous and be ordered $\zeta_1 < \ldots < \zeta_{j(C)}$.

Semi-algebraic geometry

Apply the Theorem to all the $A_{\Delta,e}$ and take $\{C_1, \ldots, C_k\}$ partition of $\mathbb{R}^n$ by semi-algebraic sets given by (iii). Let $C \in \{C_1, \ldots, C_k\}$. Then exists $\mu(C)$ such that $C \subseteq A_{\Delta,\mu(C)}$.
Claim 2: exists $j(C)$ such that

$$\#\{y \in \mathbb{R} : f_\Delta(x, y) = 0\} = j(C) \text{ for all } x \in C,$$

moreover, the $\zeta_{C,j} : C \to \mathbb{R}$ ($j = 1, \ldots, j(C)$) giving the roots are continuous and be ordered $\zeta_1 < \ldots < \zeta_{j(C)}$.

(Follows from Continuity of complex roots from complex analysis...)

Semi-algebraic geometry

Semi-algebraic geometry

Let B be one of the

$$\Gamma(\zeta_{C,j}),\ (-\infty,\zeta_{C,1}),\ (\zeta_{C,j},\zeta_{C,j+1}) \text{ and } (\zeta_{C,j(C)},+\infty).$$

Let $\epsilon_{l,r} = \mathrm{sign}(\frac{\partial^r f_l}{\partial T^r}{}_{|B})$ and let

$$B' = \{(x,t) \in C \times \mathbb{R} : \mathrm{sign}(\frac{\partial^r f_l}{\partial T^r}(x,t)) = \epsilon_{l,r},\ \forall (l,r)\}$$

<u>Then $B = B'$.</u>
If $B' \setminus B \neq \emptyset$, fix $(x,t') \in B'$ and take $(x,t) \in B$ say $t < t'$. Since $\{s \in \mathbb{R} : (x,s) \in B'\}$ is an interval (<u>Thom's lemma</u>), $[t,t'] \subseteq B'$. Contradiction: $f_{\Delta|B'}$ must change sign and it does not change sign. □

Semi-algebraic geometry

Semi-algebraic geometry

Corollary

Let $A \subseteq R^{n+1}$ be a semi-algebraic set. Then A is a finite union of semi-algebraic sets of the form

$$\Gamma(\zeta_{C,j}),\ (-\infty, \zeta_{C,1}),\ (\zeta_{C,j}, \zeta_{C,j+1}) \text{ and } (\zeta_{C,j(C)}, +\infty).$$

In particular, $\pi_i^{n+1}(A)$ is a semi-algebraic set.

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In particular, $\pi_i^{n+1}(A)$ is a semi-algebraic set.

Corollary

Let $A \subseteq R$ be definable in $(R, 0, 1, -, +, \cdot, <)$. Then A is a finite union of sets of the form $\{a\}$, $(-\infty, a), (a, b)$ or $(b, +\infty)$ with $a, b \in R$.

Semi-algebraic geometry

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Cell decomposition is a nice stratification result which gives finiteness results such as:

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Corollary (Łojasiewicz property)

Let $A \subseteq R^n$ be definable in $(R, 0, 1, -, +, \cdot, <)$. Then A has finitely many semi-algebraically connected components.

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Let $A \subseteq R^n$ be definable in $(R, 0, 1, -, +, \cdot, <)$. Then A has finitely many semi-algebraically connected components.

Corollary (Uniform Łojasiewicz property)

Let $A \subseteq R^m \times R^n$ be definable in $(R, 0, 1, -, +, \cdot, <)$. Then there is $M_A \in \mathbb{N}$ such that for each $x \in R^m$, the fiber $A_x \subseteq R^n$ has at most M_A many semi-algebraically connected components.

Sub-analytic geometry

Sub-analytic geometry

A restricted analytic function $f : \mathbb{R}^n \to \mathbb{R}$ is a function given by

$$f(\overline{x}) = \begin{cases} \sum_{\alpha \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}, & \text{for} \quad \overline{x} \in I^n \\ \\ 0, & \text{otherwise} \end{cases}$$

where $\sum_{\alpha \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is convergent in some open neighborhood of the compact box $I = [-1, 1]^n$ and $f(I^n) \subseteq I$.

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Example: restricted $\sin(x)$

$$f(x) = \begin{cases} \sum_{l\in\mathbb{N}} \frac{(-1)^l}{(2l+1)!} x^{2l+1}, & \text{for} \quad x \in I \\ 0, & \text{otherwise} \end{cases}$$

.... similarly, restricted $\cos(x)$, $\frac{1}{10}\exp(x)$, etc...

Sub-analytic geometry

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Let's find out what are the subset of $\mathbb{R}$ definable in

$$\overline{\mathbb{R}}_{\mathrm{an}} = (\mathbb{R}, 0, 1, -, +, \cdot, (f)_{f \in \mathrm{an}}, <)$$

the real field expanded by the restricted analytic functions.

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QF definable subsets of I^n are subsets of I^n semi-analytic in $\mathbb{R}^n$ i.e., finite unions of sets of the form:

$$\{\overline{x} \in I^n : \begin{cases} f_1(\overline{x}) = 0 \\ \vdots \\ f_t(\overline{x}) = 0 \end{cases} \quad \text{and} \quad \begin{cases} g_1(\overline{x}) > 0 \\ \vdots \\ g_s(\overline{x}) > 0 \end{cases} \}$$

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(... <u>local description</u> of subsets of I^n semi-analytic in $\mathbb{R}^n$....)

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The collection of subsets of I^n semi-analytic in $\mathbb{R}^n$ is closed under the operations

$$\setminus \quad \cap \quad \cup$$

What about under the operations

$$\pi_i^n?$$

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Let

$$B = \{(x, y, z, u) \in I^4 : y = xu, z = x\frac{1}{10}\exp_{|I}(u)\}$$

a compact semi-analytic in $\mathbb{R}^4$ and A the projection of B onto the first 3 coordinates. Then $A \subseteq I^3$ is not semi-analytic in $\mathbb{R}^3$.

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If it were ... since $\dim A < 3$ there would be a non zero formal power series $G(x, y, z)$ such that $G(x, xu, x\exp_{|I}(u)) = 0$. Writing $G = \sum_{j=0}^{\infty} G_j$ with G_j homogenous polynomials of degree j,

$$\sum_{j=0}^{\infty} x^j G_j(1, u, \exp_{|I}(u)) = 0$$

and so all $G_j(1, u, \exp_{|I}(u)) = 0$, and all polynomial $G_j \equiv 0$ since $\exp_{|I}(u)$ is transcendental.

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Sub-analytic geometry is the geometry of subsets sub-analytic in real-analytic spaces.

Sub-analytic geometry

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Theorem (Gabrielov's (1968))

The complement of a subset sub-analytic in a real-analytic space X is a subset sub-analytic in X.

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Theorem (Łojasiewicz (1965))

Locally a subset semi-analytic in a real-analytic space X has finitely many connected components each of which is a subset semi-analytic in X.

These things are very hard and use other stratification results (uniformization and rectilinearization) based on resolutions of singularities!

Sub-analytic geometry

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Corollary

$A \subseteq I^n$ is definable in

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Corollary

$\overline{\mathbb{R}}_{\mathrm{an}} = (\mathbb{R}, 0, 1, -, +, \cdot, (f)_{f \in \mathrm{an}}, <)$ is o-minimal.

Conclusion...

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$$\mathrm{Th}(\overline{\mathbb{R}}_{\mathrm{an}})$$

has QE if we add a function symbol $^{-1}$ for $x \mapsto \frac{1}{x}$ where $0^{-1} = 0$ by convention.

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THANK YOU!