Worms, Ordinal Worms and Ordinal Analysis

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Gödel-Löb Polymodal logic **GLP** is a provability logic that has for each ordinal α a modality $[\alpha]$, whose intended interpretation is a provability predicate in a hierarchy of theories of increasing strength. The logic $\mathbf{GLP}_\omega$ -that only has modalities $[n]$ for $n < \omega$- was first introduced by Japaridze, and recently applied by Beklemishev to give a Π^0_1-ordinal analysis of Peano Arithmetic and related systems. This ordinal analysis was carried out within the *closed fragment* of $\mathbf{GLP}_\omega$. Within this fragment, we find some particular terms; formulas of the form $\langle n_0\rangle \ldots \langle n_j\rangle\top$ -so called *worms*- that constitute an alternative ordinal notation system for ordinals below ϵ_0.

Another interesting property of $\mathbf{GLP}_\omega$ is that we have arithmetical completeness for a wide range of interpretations of $[n]$. In particular, $\mathbf{GLP}_\omega$ is sound and complete when reading $[n]$ as "provable in EA together with all true Π^0_n sentences". This reading is closely related to Turing progressions, that are hierarchies of theories such that given an initial theory T, we can construct a transfinite sequence of extensions of T by iteratedly adding n-consistency statements. Nevertheless, this link between **GLP** and Turing progressions is only by approximating the progression, so that it requires many technical results.

A weaker system, called *Reflection Calculus* **RC**, was introduced by Beklemishev and Dashkov. It is much simplier than **GLP** but yet expressive enough to maintain its main proof theoretic applications as the ones mentioned above. From the point of view of modal logic, **RC** can be seen as the positive fragment of **GLP**. An advantage of going to a positive language is that we gain a more general arithmetical interpretation. Since we discard some elements as the negation, modal formulas can be interpreted as arithmetical theories rather than arithmetical sentences.

In order to get a logic which can be used to directly denote Turing progressions, positive language together with some special worms, called *ordinal worms*, seems to be appropriate. These ordinals worms are built up from a new modality $\langle\alpha, A\rangle$, where α is an ordinal and A is a worm. The intended interpretation of this new modality would be:

$$\langle\alpha, A\rangle\varphi \equiv \langle\alpha\rangle^{o(A)}\varphi \equiv \underbrace{\langle\alpha\rangle\langle\alpha\rangle\ldots}_{o(A)-times}\varphi.$$

where $o(A)$ is the ordinal corresponding to A. This way, since worms gives us a nice ordinal notation system for ordinals below ϵ_0, and positive language allows us to interpret modal formulas as theories, we can easily use them to denote transfinite levels in a progression.